MODELING BOVINE PERICARDIUM

by

Raffaella De Vita

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This thesis was presented

by

Raffaella De Vita

It was defended on

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and approved by

Michael R. Lovell, Associate Professor, Mechanical Engineering Dept.

Anne M. Robertson, Associate Professor, Mechanical Engineering Dept.

Michael S. Sacks, Associate Professor, Bioengineering Dept.

Thesis Advisor: William S. Slaughter, Associate Professor, Mechanical Engineering Dept.

ABSTRACT

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Raffaella De Vita, M.S.

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Bovine pericardium has been widely used over the last decades as bioprosthetic material due to its excellent biocompatibility. However, the durability of pericardial heart valves is still inadequate and prevents them from being the perfect heart valve substitutes. To increase their short-time performance, studying the tissue mechanical features is of prime importance. Consequently, constitutive relations need to be developed.

The purpose of this work was to analyze two different constitutive approaches used to model bovine pericardium: the phenomenological approach and the structural approach.

Phenomenological constitutive laws are formulated to fit empirical data, independent of histological considerations. Their main drawback is the variability of material parameters for different protocols for the same specimen. Thus, they do not consent to interpret the tissue's mechanical behavior. In the second chapter, some physically sound restrictions on the parameters, which appear in some forms of the exponential Fung model, are obtained by invoking the Legendre-Hadamard and the Strong Ellipticity conditions. These restrictions can validate the empirical models and can be used in the fitting procedure.

Structural constitutive equations are determined by taking into account the tissue's architecture. A structurally based constitutive law describing the tissue's mechanical response through failure behavior has been proposed in the third chapter. The model has been tested by using published experimental data and a sufficiently good fit has been obtained.

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NOMENCLATURE

$\mathbb{R}$	real numbers
a, b, c, ..., α, β, γ...	elements of $\mathbb{R}$
$\mathbb{E}^3$	Euclidean 3-space
$\mathbf{a}, \mathbf{b}, \mathbf{c}, ...$	elements of $\mathbb{E}^3$
Lin	set of all second order tensors
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ...	elements of Lin
Lin^+	set of all second order tensors with positive determinant
Sym	set of all symmetric second order tensors
Sym^+	set of all symmetric second order tensors with positive determinant
Orth	set of all orthogonal second order tensors
Orth^+	set of all orthogonal second order tensors with positive determinant
$\mathbf{I}$	identity deformation
$\mathbf{F}$	deformation gradient
$\mathbf{C}$	right Cauchy-Green deformation tensor
$\mathbf{E}$	Green-Saint Venant strain tensor
$\mathbf{P}$	first Piola-Kirchhoff stress tensor
$\mathbf{S}$	second Piola-Kirchhoff stress tensor
$\mathbf{ab}$	dyad
$\mathbf{A}^{\mathrm{T}}$	transpose of $\mathbf{A}$
$\mathbf{A}^{\text{-T}}$	inverse of $\mathbf{A}^{\mathrm{T}}$
$\mathcal{A}, \mathcal{B}$	fourth order tensors
$\mathcal{B}$	body
$\mathcal{P}$	material point of a body $\mathcal{B}$
$\mathbf{X}$	reference position of a material point
$\mathbf{x}$	current position of a material point
.	scalar product in $\mathbb{E}^3$ or tensor product in Lin
:	scalar product in Lin
det	determinant
$\forall$	for all
$\neq$	different
$\in$	belongs to

1.0 HEART VALVES

1.1 The Heart and its Valves

The heart is a muscular organ that provides oxygenated blood to the cells of our body. It consists of four chambers: the right and the left atria, the right and the left ventricles.

After circulating through the body and delivering oxygen and nutrients, blood flows to the right atrium through veins. Once this chamber is filled, it contracts and the blood is pushed down to the right ventricle through the *tricuspid valve*. Next, when the ventricle contracts, the blood is pumped to the lungs through the *pulmonary valve*. From the lungs, oxygen-rich blood enters the left atrium by pulmonary veins. The atrium fills with blood and it contracts pumping blood to the left ventricle by the *mitral valve*. Finally, oxygenated blood goes to the aorta, the largest artery of our body, through the *aortic valve*.

The heart valves may be affected by two dysfunctions: *regurgitation* and *stenosis*.

Regurgitation is a disorder that consists into the inability of the valve to completely close. Therefore, blood leaks back instead of proceeding in the direction of the flow. The regurgitated blood needs to be pumped again. The heart responds to this anomaly by enlarging the chambers since it has to contain more blood. Therefore, the heart chamber can wear out and congestive heart failure can occur.

Stenosis is a valve abnormality that involves a narrowing of the valve opening. Then, a higher pressure is needed to pump the blood through the valve. The cardiac muscle compensates by becoming thicker. However, the narrowing can increase and heart failure can occurs.

Both problems can be corrected by a surgical replacement of the valve with either mechanical or biological valves. We present different valves and analyze their advantages and disadvantages in the next sections.

Figure 1.1. Ball in Cage Valve [7].

1.2 Mechanical Heart Valves

The mechanical valves are the forms of prosthesis commonly used for replacing damaged heart valves. They are made of different materials and have various shapes [30].

The first mechanical valve used clinically is the *ball in cage*. It consists of an occluder ball of silicon rubber in a cage made of stainless steel, or solid Teflon, or Lucite. The sewing ring is made out of Teflon cloth (Fig. (1.1)). Different variants of this valve have been designed and implanted.

A second kind of mechanical valve is the *disk valve*. There are two kinds of disk valve: *single leaflet disk valve* and *bileaflet disk valve*. The single leaflet disk valve, as the name suggests, is similar to the ball in cage but it has a disk, instead of a ball, which moves up and down inside a cage with the heartbeat (Fig. (1.2)). The performance of this valve has been improved by introducing tilting disk in the cage. The bileaflet disk valve consists of two disks instead of one. It has been preferred for low thrombogenicity and higher hemodynamics (Fig. (1.3)). It is usually made of carbon pyrolite. Recently, it has been introduced a metallic ring covered by the polyester sewing ring to allow the x-ray visibility.

The main advantage of mechanical valves is their long durability. They usually last for a lifetime and they do not require replacement. Therefore, they are implanted in old and in young patients. However, since they are made of material which is not biocompatible,

Figure 1.2. Single Leaflet Disk Valve [7].

Figure 1.3. Bileaflet Disk Valve [7].

the risk of blood clots on the valves components is high and an anticoagulation therapy is necessary. This limits their use in woman who wish to have children and in patients who have bleeding disorders.

Figure 1.4. Porcine (Pig) Stented Valve [7].

1.3 Biological Heart Valves

The tissue heart valves can be made out of human tissue or animal tissue [29].

The human tissue valves are of two types: *autographs* and *homographs* (or, equivalently, *allographs*). The autograph valves are made out of the tissue from the patient who needs the heart valve replacement. The tissue can be taken from the dura mater, fascia lata, vena cava, pericardium, and peritoneum. In the homograph valves, the tissue valve is taken out from a different human donor.

Animal heart valves are also called *xenographs* or, equivalently, *hetereographs*. The tissue used is either porcine pericardium or bovine pericardium. Porcine valve consists of pig valve attached to a Dacron covered steel frame, the so-called stent. Some porcine valves have been implanted without stents in order to increase hemodynamic properties (Fig. (1.4)).

Bovine pericardium heart valves are made from cow pericardium sewed on a Dacron covered titanium stents (Fig. (1.5)).

Tissue valves, in particular bovine pericardium valves, demonstrated excellent hemodynamic performance and low thrombogenicity. They have the advantage over the mechanical ones of not requiring long term blood thinning medication. On the other hand, their durability is limited to 15-18 years and, therefore, they are implanted only in elderly patients.

Figure 1.5. Bovine (Cow) Pericardial Stented Valve [7].

1.4 Bovine Pericardial Heart Valves

Bovine pericardium is a fibrous membrane surrounding the heart and portions of blood vessels. It consists of an outer layer, the fibrous pericardium, and an inner layer, the serous pericardium. The serous pericardium consists of other two layers: the parietal layer and the visceral layer. The parietal layer separates the fibrous layer from the serous layer while the visceral layer covers the muscular wall of the heart. The tissue used to construct the leaflets of pericardial heterografts is taken from the fibrous pericardium and the parietal layer of the serous pericardium [5].

Primary tissue failure, together with calcification, are responsible for the limited durability of pericardial heart valves [33]. Indeed, formation of leaflets tears at the edge of the cloth-covered stent is the main cause of regurgitation.

To improve the durability of the heart valves substitutes, a study of the mechanical tissue failure is needed. Consequently, characterization of mechanical properties by mean of constitutive laws plays an important role in the development of bioprosthetic heart valves using pericardium.

1.5 Constitutive Models

In order to describe the complex mechanical response of bovine pericardium, phenomenological and structural constitutive laws have been adopted.

By applying the phenomenological approach, constitutive relations are formulated to fit experimental data without requiring detailed knowledge of the composition of the material. Thus, they are suitable to computational applications, they do not allow to interpret the tissues structural properties. Consequently, the material constants lack of any physical meaning. One of the most successful phenomenological model for soft tissues has been presented by Fung [8]. It guarantees reasonably good fit to experimental data. However, non-linear regression techniques for fitting data requires setting bounds on the materials parameters. To this end, constitutive inequalities have been widely used when the material is assumed to be isotropic. For incompressible and anisotropic material by using convexity properties, Walton at al. [34] have derived some necessary and sufficient conditions for the material parameters in the classical Fung model to be satisfied. In the first chapter, we use similar arguments to find restrictions on the material parameters of some specialized forms of the Fung model previously used for bovine pericardium [20],[32]. Those conditions will be helpful into the fitting process and, furthermore, they will enable to interpret the mechanical response of the material.

Many researchers have preferred the structural approach [15],[16] to the phenomenological one. It is more difficult to implement numerically but it provides insights into the mechanical role of the different tissue's components. Zioupos [36] performed uniaxial experiments on native bovine pericardium which revealed the biomechanical characteristics of the tissue. He proposed a structurally based model which sufficiently characterizes the non-linearity and anisotropy of the material. The mechanical behavior of the tissue, considered as fibrous composites, is assumed to be determined solely by fibers component. Thus, matrix contribution is neglected. Elastin fibers appear either straight or undulated forming one subset with the collagen ones in the undeformed state. The fibers are modeled such that they bear load only when stretched. It has been shown that the model can explain the different extensibility and stiffness of the tissues strips aligned along the circumferential and the axial directions as well as the increasing thinning of the axial strip. The relationship between the angular variation of the tissue strength and the fibers density distribution is assumed to be linear and failure process is not included into the model. To our knowledge, there are no reports in the literature on structural laws which includes the description of the failure process for pericardium. However, since collagen is considered to be the determinant component of the mechanical behavior of this tissue, works on ligaments and tendons failure have driven our study [17, 12]. In the third chapter a structural constitutive model enables to determine the mechanical response of the pericardium up to failure has been presented. By using small angle light scattering technique [23], the tissue angular variation has been quantified. Finally, only four physical meaningful constants appear in the model and their values have been determined by using a differential evolution algorithm.

2.0 A PHENOMENOLOGICAL MODEL

Constitutive equations are mathematical relations defining the mechanical properties of materials. The establishment of suitable constitutive relations for soft tissues is an important and difficult task. In his works, Fung strongly emphasizes the need of constitutive equations in Biomechanics:

> The most serious frustration to a biomechanics worker is usually the lack of information about the constitutive equations of living tissues [9].

Difficulties arises from the nonlinearity of the stress-strain relation and the anisotropy exhibited by these tissues.

The mechanical behavior of some tissues has been described by mathematical relations independent of the tissue's structure, with material parameters to be determined by using empirical data. Those equations are known as *phenomenological* constitutive equations.

Bovine pericardium can be modeled as an incompressible, nonlinear elastic, anisotropic material by means of a phenomenological law, the exponential Fung model. To ensure physical plausibility, some restrictions on this constitutive equation need to be imposed.

A detailed survey of constitutive inequalities has been presented by Truesdell and Noll [28]. However, in their treatise little attention has been given to constitutive equations and their static implications for anisotropic material.

In this chapter, after presenting the Strong Ellipticity and the Legendre-Hadamard inequalities with their mechanical implications, we specialized those inequalities for incompressible materials and we illustrated their static implications. We found bounds on the material constants of some forms of the exponential Fung model by following Walton and Wilber's work. Setting restrictions on the parameters of this phenomenological description is important to establish its physical plausiblility.

2.1 Order Preserving Inequalities

Let Lin denote the space of all second-order tensors and Lin^+ denote the subset of Lin consisting of second-order tensors with positive determinant. Furthermore, let Sym and Orth be the subsets of symmetric and orthogonal second-order tensors, respectively. Let Sym^+ and Orth^+ denote the second-order tensors of Sym and Orth, respectively, with positive determinant.

Let $\mathbf{X}$ be the reference position of a material point $\mathcal{P}$ of a body $\mathcal{B}$ and let $\mathbf{P}$ denote the first Piola-Kirchhoff. The material of the body $\mathcal{B}$ is elastic if there exists a function

$$\mathrm{Lin}^+ \times \mathcal{B} \ni (\mathbf{F}, \mathbf{X}) \longmapsto \hat{\mathbf{P}}(\mathbf{F}, \mathbf{X}) \equiv \mathbf{F} \cdot \hat{\mathbf{S}}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X}), \tag{2.1}$$

with $\hat{\mathbf{S}}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X}) \in \mathrm{Sym}$ such that at any time t

$$\mathbf{P}(\mathbf{X}, t) = \hat{\mathbf{P}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}), \tag{2.2}$$

where $\mathbf{F}$ is the deformation gradient (Note that $\hat{\mathbf{S}}$ needs to be an element of Sym to satisfy the balance of angular momentum and W needs to be a function of $\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ to satisfy the principle of frame-indifference). The function $\hat{\mathbf{P}}$ is called a *constitutive equation* and describes the mechanical properties of the body.

We briefly present some restrictions on the constitutive function $\hat{\mathbf{P}}$ which will be imposed on particular constitutive relations in the next section.

An elastic material pulled on in one direction elongates in the same direction (Fig.(2.1)). This obvious physical observation is not easily translated into a precise and universally valid mathematical statement since there are many kinds of stresses to measure the amount of pull, many kinds of strains to measure the amount of elongation and, in addition, when the material is pulled on in one direction, it must contracts in the transverse direction. However, in order to express mathematically the previous physical observation, it seems reasonable to demand $\hat{\mathbf{P}}(\cdot, \mathbf{X})$ to be *strictly monotone.*

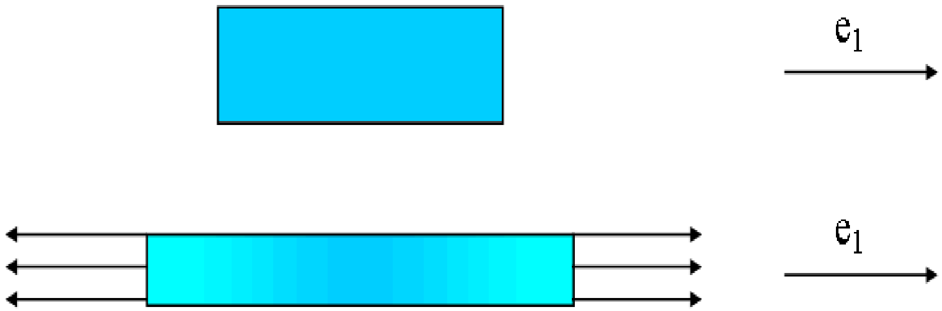

Figure 2.1. An elastic material pulled in the $\mathbf{e}_1$ direction elongates in the $\mathbf{e}_1$ direction.

Let ":" denote the inner product in Lin. We note that the function $\hat{\mathbf{P}}(\cdot, \mathbf{X})$ is defined on the nonconvex set[1]. Thus, $\hat{\mathbf{P}}(\cdot, \mathbf{X})$ is defined to be strictly monotone in Lin^+ if the restriction of $\hat{\mathbf{P}}$ to each line segment of Lin^+ is strictly monotone. Therefore, $\hat{\mathbf{P}}(\cdot, \mathbf{X})$ is strictly monotone if

$$[\hat{\mathbf{P}}(\mathbf{G} + \alpha\mathbf{H}, \mathbf{X}) - \hat{\mathbf{P}}(\mathbf{G}, \mathbf{X})] : \mathbf{H} > 0 \quad \forall\, \mathbf{G} \in \mathrm{Lin}^+, \tag{2.3}$$

$$\forall\, \mathbf{H} \neq \mathbf{0}, \quad \forall\, \alpha \in (0, 1] \quad \text{such that} \quad \det(\mathbf{G} + \alpha\mathbf{H}) > 0. \tag{2.4}$$

Assume $\hat{\mathbf{P}}(\cdot, \mathbf{X})$ to be differentiable, i.e. by definition assume that there exists a fourth-order tensor $\frac{\partial \hat{\mathbf{P}}}{\partial \mathbf{F}}(\mathbf{F}, \mathbf{X})$ such that

$$\lim_{\alpha \to 0} \frac{[\hat{\mathbf{P}}(\mathbf{F} + \alpha\mathbf{H}, \mathbf{X}) - \hat{\mathbf{P}}(\mathbf{F}, \mathbf{X})]}{\alpha} = \frac{\partial \hat{\mathbf{P}}}{\partial \mathbf{F}}(\mathbf{F}, \mathbf{X}) : \mathbf{H}. \tag{2.5}$$

Then, by dividing (2.3) by α and by taking the limit as $\alpha \to 0$, (2.3)-(2.4) take the following form

$$\mathbf{H} : \frac{\partial \hat{\mathbf{P}}}{\partial \mathbf{F}}(\mathbf{F}, \mathbf{X}) : \mathbf{H} > 0 \quad \forall\, \mathbf{F} \in \mathrm{Lin}^+, \quad \forall\, \mathbf{H} \neq \mathbf{0}. \tag{2.6}$$

[1] A subset of Lin is *convex* if, whenever it contains two points $\mathbf{A}$ and $\mathbf{B}$, it contains the closed segment $\mathbf{A} + \alpha\mathbf{B}$ with $\alpha \in [0, 1]$. A subset of Lin, which is not convex, is said to be *nonconvex*. It can be easily seen that the subset Lin^+ of Lin is nonconvex since if $\mathbf{A}$, $\mathbf{B} \in \mathrm{Lin}^+$, then the closed segment $\mathbf{A} + \alpha\mathbf{B} \notin \mathrm{Lin}^+$ with $\alpha \in [0, 1]$

Recall that an elastic material is said to be *hyperelasic* if there exists a nonegative scalar-valued function, called *strain energy function*, $W(\cdot, \mathbf{X})$:

$$\mathrm{Lin}^+ \ni \mathbf{F} \to W(\mathbf{F}, \mathbf{X}) = \tilde{W}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X}), \tag{2.7}$$

such that

$$\hat{\mathbf{P}}(\mathbf{F}, \mathbf{X}) = \frac{\partial \tilde{W}}{\partial \mathbf{F}}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X}). \tag{2.8}$$

Furthermore, recall that a function $W(\cdot, \mathbf{X})$ is said to be *strictly convex* in Lin^+ if the restriction of $W(\cdot, \mathbf{X})$ to each line segment of Lin^+ is strictly convex, i.e.

$$W(\lambda(\mathbf{G} + \alpha\mathbf{H}) + (1-\lambda)\mathbf{G}, \mathbf{X}) < \lambda W(\mathbf{G} + \alpha\mathbf{H}, \mathbf{X}) + (1-\lambda) W(\mathbf{G}, \mathbf{X}), \tag{2.9}$$

$$\forall\, \mathbf{G} \in \mathrm{Lin}^+,\ \forall \mathbf{H} \neq \mathbf{0},\ \forall \alpha \in (0,1] \text{ such that } \det(\mathbf{G} + \alpha\mathbf{H}) > 0,\ \forall \lambda \in (0,1). \tag{2.10}$$

In addition, note that a necessary and sufficient condition for $W(\cdot, \mathbf{X})$ to be strictly convex is $\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X})$ is strictly increasing in Lin^+.

Thus, for an hyperelastic material with strain energy function $W(\cdot, \mathbf{X})$, (2.6) is equivalent to require that

$$\mathrm{Lin}^+ \ni \mathbf{F} \to W(\mathbf{F}, \mathbf{X}) = \tilde{W}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \mathbf{X}) \quad \text{is } \textit{strictly convex.} \tag{2.11}$$

However, restrictions (2.3)-(2.4) have been rejected for mainly three reasons. First, they guarantee existence of “weak solutions”and uniqueness (to within rigid displacement) of boundary-value problems for the equilibrium equation. Therefore, because of uniqueness, a rod cannot buckle under thrust no matter how thin is the rod and how large is the thrust.

Second, they discord with the physically reasonable condition that infinite stresses are necessary to maintain extreme strains, strains for which $|\mathbf{C}| = \infty$ or $\det \mathbf{C} = 0$. The reader is referred to [4] for more details.

Finally, if the material is stress-free at the identity deformation, (2.3)-(2.4) are incompatible with the principle of frame-indifference. To prove it, let assume that

$$\hat{\mathbf{P}}(\mathbf{I}, \mathbf{X}) = \mathbf{0}, \tag{2.12}$$

and that (2.3)-(2.4) hold. The Principle of Frame-indifference states that

$$\hat{\mathbf{P}}(\mathbf{Q}\cdot\mathbf{F},\mathbf{X}) = \mathbf{Q}\cdot\hat{\mathbf{P}}(\mathbf{F},\mathbf{X}) \quad \forall\mathbf{Q}\in \mathrm{Orth}^+, \quad \forall\mathbf{F}\in \mathrm{Lin}^+ \tag{2.13}$$

By taking $\mathbf{G}=\mathbf{I}$ into (2.13) and by using (2.12), it follows that

$$\hat{\mathbf{P}}(\mathbf{Q},\mathbf{X}) = 0 \quad \forall\mathbf{Q}\in \mathrm{Orth}^+. \tag{2.14}$$

Next, let choose $\mathbf{F}=\mathbf{I}$, $\mathbf{H}=\frac{1}{\alpha}(\mathbf{Q}-\mathbf{I})$ into (2.3)-(2.4). Then, it follows that

$$\hat{\mathbf{P}}(\mathbf{Q},\mathbf{X}):(\mathbf{Q}-\mathbf{I})>0, \tag{2.15}$$

which contradicts $\hat{\mathbf{P}}(\mathbf{Q},\mathbf{X})=\mathbf{0}$.

Due to the above-mentioned reasons, a weaker condition known as *Strong Ellipticity Condition* has been preferred. Its statical implications are laid down by Hayes [10]. The reader must be warned that, although attractive, this assumption is not comprehensive. Ericksen has showed that Strong Ellipticity Condition fails when phase transitions occurs. The Strong Ellipticity condition can be expressed as follows

$$[\hat{\mathbf{P}}(\mathbf{G}+\alpha\mathbf{ab},\mathbf{X})-\hat{\mathbf{P}}(\mathbf{G},\mathbf{X})]:\mathbf{ab}>0 \quad \forall\,\mathbf{G}\in\ \mathrm{Lin}^+, \tag{2.16}$$

$$\forall\,\mathbf{a},\mathbf{b}\in\mathbb{E}^3\setminus\{\mathbf{0}\}\,,\forall\,\alpha\in(0,1] \quad \text{such that} \quad \det(\mathbf{G}+\alpha\mathbf{ab})>0. \tag{2.17}$$

where $\mathbb{E}^3$ denotes the Euclidean 3-space. Using the letters S-E to recall Strong Ellipticity, we refer to (2.16)-(2.17) as the *S-E condition*.

As before, when $\hat{\mathbf{P}}(\cdot,\mathbf{X})$ is differentiable, the restriction (2.16)-(2.17) becomes

$$\mathbf{ab}:\frac{\partial\hat{\mathbf{P}}}{\partial\mathbf{F}}(\mathbf{F},\mathbf{X}):\mathbf{ab}>0 \quad \forall\,\mathbf{F}\in\mathrm{Lin}^+, \quad \forall\mathbf{a},\mathbf{b}\in\mathbb{E}^3\setminus\{\mathbf{0}\}. \tag{2.18}$$

Replacing the strict inequality in (2.18) with the weak inequality gives the *Legendre-Hadamard Condition*, which will be called *L-H condition*.

2.1.1 Mechanical Interpretation of the S-E and L-E condition

Let $\mathbf{a}$ and $\mathbf{b}$ be independent of $\mathbf{F}$, or more generally, let $\mathbf{a}$ and $\mathbf{b}$ be independent of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$, then (2.18) assumes the form

$$\frac{\partial(\mathbf{a}\cdot\mathbf{P}\cdot\mathbf{b})}{\partial(\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b})} > 0, \tag{2.19}$$

Obviously, L-H condition take the form (2.19) with the strict inequality replaced by the weak inequality. Define the following set

$$\mathcal{D}(\mathbf{ab}) = \{\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b} \in \mathbb{R} : \det\mathbf{F} > 0\}. \tag{2.20}$$

Let show that $\mathcal{D}(\mathbf{ab})$ defines an open half line, or a line, or the empty set in $\mathbb{R}$. To this end, let decompose $\mathbf{F}$ along $\mathbf{ab}$ and along the orthogonal complement of $\mathbf{ab}$:

$$\mathbf{F} = (\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b})\mathbf{ab} + [\mathbf{F} - (\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b})\mathbf{ab}]. \tag{2.21}$$

If $\mathbf{a}$ and $\mathbf{b}$ are constant, then

$$\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b} \to \det\mathbf{F} \quad \text{is affine}, \tag{2.22}$$

since the determinant is an affine function of each of its entries. Hence, $\mathcal{D}(\mathbf{ab})$ is an open half line, or an open line, or the empty set. Particularly, if the cofactor of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$ is not zero, then $\mathcal{D}(\mathbf{ab})$ is a half line.

If $\mathbf{a}$ and $\mathbf{b}$ are independent of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$, then $\mathcal{D}(\mathbf{ab})$ is still an open half line, or a line, or the empty set. To prove this, let $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_j\}$ be right handed orthonormal bases for $\mathbb{E}^3$ with $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{b}_1 = \mathbf{b}$ and let $\mathbf{F} = F_{ij}\mathbf{a}_i\mathbf{b}_j$. Note that $\det\mathbf{F} \equiv [(\mathbf{F}\cdot\mathbf{b}_1)\times(\mathbf{F}\cdot\mathbf{b}_2)]\cdot(\mathbf{F}\cdot\mathbf{b}_3) = e_{ijk}F_{i1}F_{j2}F_{k3}$ with e_{ijk} is the permutation symbol. The variables F_{ij} with $(i,j)\neq(1,1)$ are independent of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$ since $\mathbf{a}$ and $\mathbf{b}$ are independent of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$. It follows that $\det\mathbf{F}$ is still affine in $F_{11} = \mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$. Thus, we can rewrite

$$\mathcal{D}(\mathbf{ab}) = (l^-(\mathbf{ab}), l^+(\mathbf{ab})), \tag{2.23}$$

where $l^{-}(\mathbf{ab})$ is either $-\infty$ or a finite number and $l^{+}(\mathbf{ab})$ is either ∞ or a finite number. Therefore, inequality (2.19) implies that

$$\mathcal{D}(\mathbf{ab}) \ni \mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b} \rightarrow \mathbf{a}\cdot\mathbf{P}\cdot\mathbf{b} \quad \text{is strictly increasing.} \tag{2.24}$$

Thus, the component $\mathbf{a}\cdot\mathbf{P}\cdot\mathbf{b}$ of the first Piola-Kirchhoff is a strictly increasing function of the corresponding component $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{b}$ of the deformation gradient. Note that such function is an increasing function if the L-H condition is satisfied.

2.1.2 S-E and L-H Conditions for Incompressible Elastic Material

We now specialize the S-E condition and the L-H condition for incompressible elastic media since such bovine pericardium is assumed to be.

Let $\mathbf{a},\mathbf{b} \in \mathbb{E}^3$ and let $\hat{\mathbf{P}}_A$ be the *active part* of the first Piola Kirchhoff stress tensor [1]. For the S-E condition we require that [1]

$$[\hat{\mathbf{P}}_A(\mathbf{G}+\alpha\mathbf{ab},\mathbf{X}) - \hat{\mathbf{P}}_A(\mathbf{G},\mathbf{X})] : \mathbf{ab} > 0, \quad \forall\, \mathbf{G}\ \in \mathrm{Lin}^{+}, \tag{2.25}$$

$$\forall\, \mathbf{ab} \neq \mathbf{0}, \quad \forall\, \alpha \in (0,1] \quad \text{such that} \quad (\mathbf{G}+\alpha\mathbf{ab})^{-\mathrm{T}} : \mathbf{ab} = 0. \tag{2.26}$$

As previously noted, if the constitutive function $\hat{\mathbf{P}}(\cdot,\mathbf{X})$ is differentiable, then (2.25)-(2.26) become

$$\mathbf{ab} : \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F},\mathbf{X}) : \mathbf{ab} > 0 \quad \forall\, \mathbf{F} \in \mathrm{Lin}^{+} \tag{2.27}$$

$$\forall\, \mathbf{ab} \neq \mathbf{0} \quad \text{such that} \quad \mathbf{F}^{-\mathrm{T}} : \mathbf{ab} = 0. \tag{2.28}$$

For incompressible material the L-H can be expressed as (2.27)-(2.28) with the strict inequality replaced by the weak inequality. Those two conditions will be extensively applied in the next sections.

2.1.3 Static Implication of the S-E and L-E conditions for Incompressible Materials

In order to investigate a static implication of the S-E *condition* and L-H *condition* for incompressible elastic material, let consider a simple shear of amount ϵ defined as

$$\mathbf{x} = [\mathbf{1} + \epsilon(\mathbf{ab})] \cdot \mathbf{X} \tag{2.29}$$

where $\mathbf{X}$ and $\mathbf{x}$ denote the reference and the current position vectors of a particle of the body $\mathcal{B}$, respectively, and $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$ are orthonormal vectors. Thus, the gradient of deformation $\tilde{\mathbf{F}}$ has the form

$$\tilde{\mathbf{F}} = \mathbf{1} + \epsilon(\mathbf{ab}), \tag{2.30}$$

and, whence,

$$\det \tilde{\mathbf{F}} = 1, \qquad \tilde{\mathbf{F}}^{-\mathrm{T}} : \mathbf{ab} = 0, \qquad \frac{\partial \tilde{\mathbf{F}}}{\partial \epsilon} = \mathbf{ab}. \tag{2.31}$$

Hence, substituting $(2.31)_3$ into (2.27) gives

$$\frac{\partial[\mathbf{a} \cdot \hat{\mathbf{P}}_A(\tilde{\mathbf{F}}) \cdot \mathbf{b}]}{\partial \epsilon} > 0, \tag{2.32}$$

where the quantity $\mathbf{a} \cdot \hat{\mathbf{P}}_A(\tilde{\mathbf{F}}) \cdot \mathbf{b}$ is the shear stress in the direction of the shear.

We conclude from (2.32) that, if the S-E condition is satisfied, any shear stress associated with the simple shear of an elastic body must be a strictly increasing function (an increasing function when the L-H condition is assumed to hold) of the amount of the shear [10].

2.2 L-H and S-E Conditions for Fung Model

2.2.1 Fung Model

As mentioned in chapter 1 section (1.4), a phenomenological constitutive law used to model the response of bovine pericardium is the Fung model [8] in its various formulations [32].

Assuming the material to be hyperelastic and incompressible, the first Piola-Kirchhoff stress tensor $\mathbf{P}$ is given by

$$\mathbf{P} = -p\mathbf{F}^{-\mathrm{T}} + \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}} \tag{2.33}$$

where p is the pressure enforcing incompressibility and $\hat{\mathbf{P}}_A = \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}}$ is the *active part* of the first Piola Kirchhoff stress tensor [1].

The *strain energy function* W has the form

$$W = \frac{c}{2}(e^Q - 1), \tag{2.34}$$

with c positive constant and Q defined by

$$Q = \mathbf{E} : \mathcal{A} : \mathbf{E} \tag{2.35}$$

where $\mathbf{E} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{C} - \mathbf{I})$ is the *material* or *Green-Saint Venant strain tensor* and $\mathcal{A}$ is a constant fourth-order tensor.

The model (2.34)-(2.35) has been shown to fit experimental data sufficiently well with only 4 and 6 nonzero components of $\mathcal{A}$ [20]. The following simplified forms of Q, Q_1 and Q_2, have been adopted

$$Q_1 = A_1 E_{11}^2 + A_2 E_{22}^2 + 2A_3 E_{11} E_{22} + A_4 E_{12}^2, \tag{2.36}$$

$$Q_2 = A_1 E_{11}^2 + A_2 E_{22}^2 + 2A_3 E_{11} E_{22} + A_4 E_{12}^2 + 2A_5 E_{11} E_{12} + 2A_6 E_{12} E_{22}. \tag{2.37}$$

The goal of this section is to derive some restrictions on the material parameters A_1, A_2, A_3, A_4, A_5, and A_6 by requiring the S-E or the L-H condition to be satisfied.

2.2.2 Restrictions on Material Parameters of W with $Q = Q_1$

Consider the strain energy function (2.34) with $Q = Q_1$. Let $\{\mathbf{e}_i\}$ be an orthonormal basis for $\mathbb{E}^3$. Therefore, $\{\mathbf{e}_i\mathbf{e}_j\}$ is a basis for Lin and $\{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\}$ is a basis for the space of the fourth-order tensors. Then, the fourth-order tensor $\mathcal{A}$ of (2.35) can be written in this basis as

$$\begin{aligned}\mathcal{A}_1 &= A_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 + A_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + A_3(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1)\\ &\quad + \frac{A_4}{4}(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1),\end{aligned} \tag{2.38}$$

and, consequently,

$$Q_1 = \mathbf{E} : \mathcal{A}_1 : \mathbf{E}. \tag{2.39}$$

Since the constitutive function $\hat{\mathbf{P}}_A$ is differentiable and bovine pericardium is modeled as an incompressible material, we apply the S-E and the L-H condition for incompressible material to impose some restrictions on A_1, A_2, A_3, and A_4 obtaining the following result:

Theorem 2.2.1 *If the Fung model (2.34) with $Q = Q_1$ satisfies the Legendre-Hadamard condition, then the parameters A_1, A_2, A_3, A_4 are non-negative and they satisfy*

$$\sqrt{A_2\left(A_3 + \frac{A_4}{2}\right)} \geq \frac{1}{2}(A_2 + A_3), \tag{2.40}$$

$$\sqrt{A_1\left(A_3 + \frac{A_4}{2}\right)} \geq \frac{1}{2}(A_1 + A_3). \tag{2.41}$$

Proof: By using the symmetries of the fourth-order tensor $\mathcal{A}_1$, it has been proved [34] that $\forall\, \mathbf{H} \in$ Lin,

$$\begin{aligned}\frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} &= c\, e^{Q_1}[(\mathbf{E} : \mathcal{A}_1 : (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{H}^{\mathrm{T}} \cdot \mathbf{F})) \cdot \mathbf{F} \cdot (\mathcal{A}_1 : \mathbf{E}) + \mathbf{H} \cdot (\mathcal{A}_1 : \mathbf{E})]\\ &\quad + \frac{1}{2}\mathbf{F} \cdot (\mathcal{A}_1 : (\mathbf{F} \cdot \mathbf{H} + \mathbf{H}^{\mathrm{T}} \cdot \mathbf{F}))],\end{aligned} \tag{2.42}$$

and, whence,

$$\mathbf{H} : \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} = c\, e^{Q_1}[2(\mathbf{E} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + \mathbf{E} : \mathcal{A}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H}]. \tag{2.43}$$

Then, condition the L-H condition for incompressible material is verified when

$$2(\mathbf{E} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + \mathbf{E} : \mathcal{A}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} \geq 0 \quad \forall\, \mathbf{F} \in \mathrm{Lin}^+, \tag{2.44}$$

$$\forall\, \mathbf{H} = \mathbf{ab} \neq \mathbf{0} \quad \text{with} \quad \mathbf{a}, \mathbf{b} \in \mathbb{E}^3 \quad \text{such that} \quad \mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0. \tag{2.45}$$

Let take $\mathbf{H} = \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Thus, we obtain that

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \lambda_1\mathbf{e}_1\mathbf{e}_2, \quad \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_2\mathbf{e}_2, \quad \mathbf{E} = \frac{1}{2}\{(\lambda_1^2 - 1)\mathbf{e}_1\mathbf{e}_1 + (\lambda_2^2 - 1)\mathbf{e}_2\mathbf{e}_2\}. \tag{2.46}$$

It follows that

$$\mathbf{E} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = 0, \tag{2.47}$$

$$\mathbf{E} : \mathcal{A}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \frac{1}{2}\{(\lambda_2^2 - 1)A_2 + (\lambda_1^2 - 1)A_3\}, \tag{2.48}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \frac{A_4}{4}\lambda_1^2. \tag{2.49}$$

Hence, requiring (2.44)-(2.45) to be satisfied for the previous choices of $\mathbf{H}$ and $\mathbf{F}$ is equivalent to demand the following inequality to hold

$$2[(\lambda_2^2 - 1)A_2 + (\lambda_1^2 - 1)A_3] + A_4\lambda_1^2 \geq 0. \tag{2.50}$$

For incompressibility, $\det \mathbf{F} = \lambda_1\lambda_2 = 1$. Thus, setting $\lambda_2^2 := x$ implies that $\lambda_1^2 = \frac{1}{x}$. Hence, equation (2.50) reduces to

$$g(x) := 2xA_2 + \frac{1}{x}\left(2A_3 + A_4\right) - 2(A_2 + A_3) \geq 0 \quad \forall x. \tag{2.51}$$

Taking x very large or close to zero in (2.51) implies

$$A_2 > 0 \quad \text{and} \quad 2A_3 + A_4 > 0. \tag{2.52}$$

Next, consider $g''(x)$. Assuming $(2.52)_2$ to hold, we find that

$$g''(x) = \frac{2}{x^3}\Big(2A_3 + A_4\Big) > 0 \quad \forall x. \tag{2.53}$$

Thus, $g(x)$ is a strictly convex function in $(0,\infty)$ and, hence, it has an unique minimum on this interval. In order to find the extreme x_{min} of $g(x)$, we consider

$$g'(x) = 2A_2 - \frac{1}{x^2}\Big(2A_3 + A_4\Big), \tag{2.54}$$

and, we note

$$g'(x) = 0 \quad \text{at} \quad x_{min} = \sqrt{A_3 + \frac{A_4}{2} A_2}. \tag{2.55}$$

Hence, the minimum of $g(x)$, g_{min}, is given by

$$g_{min} = g(x_{min}) = 4\sqrt{A_2\Big(A_3 + \frac{A_4}{2}\Big)} - 2(A_2 + A_3). \tag{2.56}$$

In virtue of (2.51), we easily deduce (2.40) from (2.56).

Similarly, let set $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Then, it follows

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \lambda_2\mathbf{e}_2\mathbf{e}_1, \quad \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_1\mathbf{e}_1, \quad \mathbf{E} = \frac{1}{2}\{(\lambda_1^2 - 1)\mathbf{e}_1\mathbf{e}_1 + (\lambda_2^2 - 1)\mathbf{e}_2\mathbf{e}_2\}. \tag{2.57}$$

Thus, we obtain

$$\mathbf{E} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = 0, \tag{2.58}$$

$$\mathbf{E} : \mathcal{A}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \frac{1}{2}\{(\lambda_1^2 - 1)A_1 + (\lambda_2^2 - 1)A_3\}, \tag{2.59}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \frac{A_4}{4}\lambda_2^2. \tag{2.60}$$

Therefore, (2.44)-(2.45) are satisfied for the given $\mathbf{H}$ and $\mathbf{F}$ if and only if

$$2[(\lambda_1^2 - 1)A_1 + (\lambda_2^2 - 1)A_3] + A_4\lambda_1^2 \geq 0. \tag{2.61}$$

Because of incompressibility, setting $x := \lambda_1^2$ implies that $\lambda_2^2 = \frac{1}{x}$. Hence, (2.61) is satisfied if

$$f(x) := 2xA_1 + \frac{1}{x}\Big(2A_3 + A_4\Big) - 2(A_1 + A_3) \geq 0 \quad \forall x. \tag{2.62}$$

By taking x very large or close to zero in (2.62), we obtain

$$A_1 > 0 \quad \text{and} \quad 2A_3 + A_4 > 0. \tag{2.63}$$

Consider $f''(x)$. Note that in virtue of $(2.63)_2$,

$$f''(x) = \frac{2}{x^3}\Big(2A_3 + A_4\Big) > 0 \quad \forall x. \tag{2.64}$$

The function $f(x)$ is strictly convex in $(0, \infty)$ and, therefore, has an unique minimum on this interval. To evaluate the extreme x_{min} of $f(x)$ we consider

$$f'(x) = 2A_1 - \frac{1}{x^2}\Big(2A_3 + A_4\Big), \tag{2.65}$$

and we find

$$f'(x) = 0 \quad \text{at} \quad x_{min} = \sqrt{\frac{A_3 + \frac{A_4}{2}}{A_1}}. \tag{2.66}$$

Thus, we obtain

$$f_{min} = f(x_{min}) = 4\sqrt{A_1\Big(A_3 + \frac{A_4}{2}\Big)} - 2(A_1 + A_3). \tag{2.67}$$

By using (2.62)-(2.67), we deduce (2.41).

Theorem 2.2.2 *If the Fung model (2.34) with $Q = Q_1$ satisfies the Strong Ellipticity condition, then the parameters A_1, A_2, A_3, A_4 are positive and they satisfy*

$$\sqrt{A_2\Big(A_3 + \frac{A_4}{2}\Big)} > \frac{1}{2}(A_2 + A_3), \tag{2.68}$$

$$\sqrt{A_1\Big(A_3 + \frac{A_4}{2}\Big)} > \frac{1}{2}(A_1 + A_3). \tag{2.69}$$

Proof: We omit the proof since it follows from the same arguments of theorem (2.2.1).

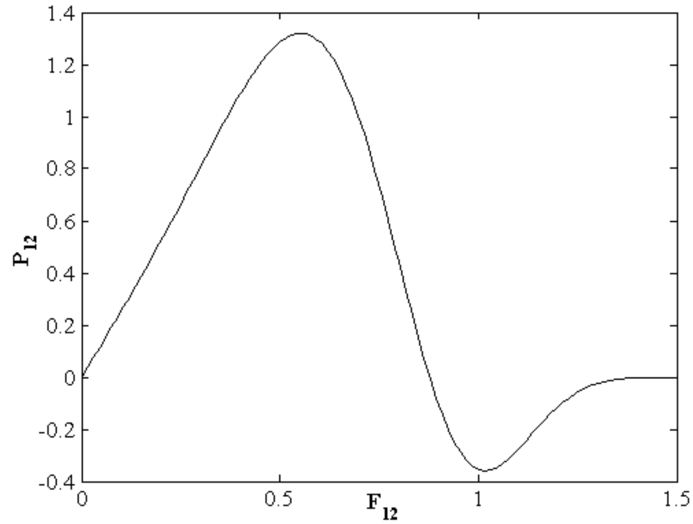

Figure 2.2. P_{12} is not an increasing function of F_{12} ($A_2 = -13$, $A_4 = -10$).

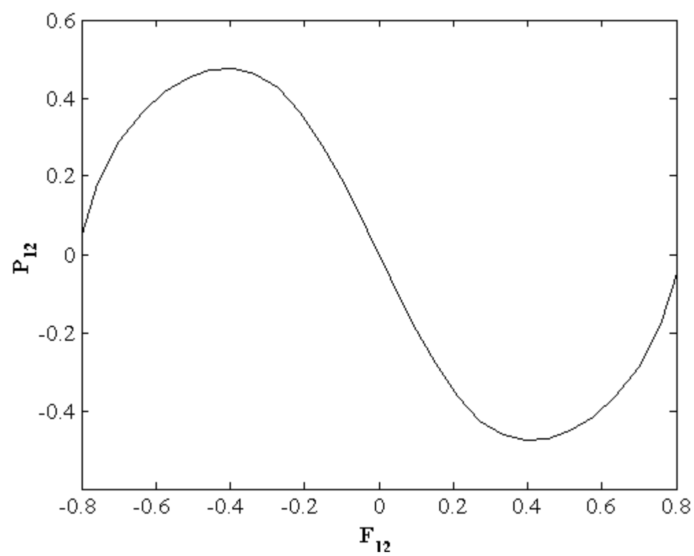

Figure 2.3. P_{12} is not an increasing function of F_{12} ($A_2 = 12$, $A_4 = -8$).

Remark: To obtain restrictions on the material parameters in the Fung model (2.34) with $Q = Q_1$, we impose the S-E condition or the L-H condition to be satisfied at some particular deformation $\mathbf{F}$ and for some $\mathbf{H} = \mathbf{ab}$ such that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Hence, those restrictions are only necessary for the S-E and L-H conditions. In other words, if the restrictions on the parameters are not satisfied, we can claim that surely the S-E and the L-H conditions are not verified. If those restrictions are verified, then we cannot say anything about the S-E and the L-H conditions.

We note that if $c = 0.5$, $A_2 = -13$, $A_4 = 10$ so that the restrictions found in theorems (2.2.1)-(2.2.2) are not satisfied, the L-H condition and the S-E condition do not hold at $\mathbf{F} = \mathbf{I} + \epsilon \mathbf{e}_1\mathbf{e}_2$. Hence, P_{12} happens to be not an increasing function of F_{12} as Fig.(2.2) shows. Similarly, Fig.(2.3) shows that, when $c = 0.5$, $A_2 = 12$, $A_4 = -8$, P_{12} is not an increasing function of F_{12}.

2.2.3 Restrictions on Material Parameters of W with $Q = Q_2$

Let consider the strain energy function (2.34) with $Q = Q_2$. Then, the fourth-order tensor $\mathcal{A}$ in (2.35) has the form

$$\begin{aligned}\mathcal{A}_2 \;=\;& A_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 + A_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + A_3(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1) + \frac{A_4}{4}(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 \\ &+ \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1) + \frac{A_5}{2}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1) \\ &+ \frac{A_6}{2}(\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2),\end{aligned} \tag{2.70}$$

in the basis $\{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\}$. Thus, Q_2 can be written as

$$Q_2 = \mathbf{E} : \mathcal{A}_2 : \mathbf{E}. \tag{2.71}$$

In order to restrict the range of the material parameters A_1, A_2, A_3, A_4, A_5 and A_6, we require the L-H condition to hold. We obtain the following result:

Theorem 2.2.3 *If the Fung model (2.34) with $Q = Q_2$ satisfies the Legendre-Hadamard condition, the parameters A_1, A_2, A_3, A_4 are non-negative and satisfy*

$$\sqrt{A_2\left(A_3 + \frac{A_4}{2}\right)} \geq \frac{1}{2}(A_2 + A_3), \tag{2.72}$$

$$\sqrt{A_1\left(A_3 + \frac{A_4}{2}\right)} \geq \frac{1}{2}(A_1 + A_3), \tag{2.73}$$

$$A_4 \geq \frac{3A_5^2}{2A_1}, \tag{2.74}$$

$$A_4 \geq \frac{3A_6^2}{2A_2}. \tag{2.75}$$

Proof: By simply using symmetries of the fourth-order tensor $\mathcal{A}_2$, a straightforward computation [34] shows that $\forall\, \mathbf{H} \in \mathrm{Lin}$,

$$\begin{aligned} \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} \;\; &= \;\; c\, e^{Q_1}[(\mathbf{E} : \mathcal{A}_2 : (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{H}^{\mathrm{T}} \cdot \mathbf{F})) \cdot \mathbf{F} \cdot (\mathcal{A}_2 : \mathbf{E}) + \mathbf{H} \cdot (\mathcal{A}_2 : \mathbf{E})] \\ &\quad + \frac{1}{2}\mathbf{F} \cdot (\mathcal{A}_2 : (\mathbf{F} \cdot \mathbf{H} + \mathbf{H}^{\mathrm{T}} \cdot \mathbf{F}))], \end{aligned} \tag{2.76}$$

and, consequently,

$$\mathbf{H} : \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} = c e^{Q_1}[2(\mathbf{E} : \mathcal{A}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + \mathbf{E} : \mathcal{A}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H}]. \tag{2.77}$$

As proved in theorem (2.2.1), by choosing $\mathbf{H} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$, if the L-H condition hold then A_1, A_2, A_3, A_4 are non-negative and (2.72)-(2.73) are verified.

Note that a necessary condition for the L-H condition to hold is

$$\mathbf{E} : \mathcal{A}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} \geq 0 \quad \forall\, \mathbf{F} \in \mathrm{Lin}^+, \tag{2.78}$$

$$\forall\, \mathbf{H} = \mathbf{ab} \neq \mathbf{0} \quad \text{with} \quad \mathbf{a}, \mathbf{b} \in \mathbb{E}^3 \quad \text{such that} \quad \mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0. \tag{2.79}$$

In order to derive restrictions on A_5, we take $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \mathbf{I} + k\mathbf{e}_2\mathbf{e}_1$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Therefore, it follows that

$$\mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_1\mathbf{e}_1, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = k\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1, \quad \mathbf{E} = \frac{k^2}{2}\mathbf{e}_1\mathbf{e}_1 + \frac{k}{2}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) \tag{2.80}$$

and, whence,

$$\mathbf{E} : \mathcal{A}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = A_1 \frac{k^2}{2} + A_5 \frac{k}{2}, \tag{2.81}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = A_1 k^2 + A_5 k + \frac{A_4}{4}. \tag{2.82}$$

Then, inequality (2.78) with (2.79) becomes

$$p(k) := 6(A_1 k^2 + A_5 k) + A_4 \geq 0 \quad \forall k. \tag{2.83}$$

Taking k very large in (2.83) implies that $A_1 > 0$. Thus, let consider $p''(k)$ and note that

$$p''(k) = 12 A_1 > 0 \quad \forall k. \tag{2.84}$$

It follows that $p(k)$ is a strictly convex function in $(-\infty, \infty)$ and, hence, it has a unique minimum. To find the extreme k_{min} of $p(k)$, we evaluate

$$p'(k) = 6(2A_1 k + A_5), \tag{2.85}$$

and, immediately, we obtain

$$p'(k) = 0 \quad \text{at} \quad k_{min} = -\frac{A_5}{2A_1}. \tag{2.86}$$

The minimum of $p(k)$, p_{min}, is given by

$$p_{min} = p(k_{min}) = -\frac{3A_5^2}{2A_1} + A_4. \tag{2.87}$$

In virtue of (2.83)-(2.87), we obtain (2.74).

We apply similar arguments to find restrictions on A_6. Thus, we take $\mathbf{H} = \mathbf{e}_1\mathbf{e}_2$, and $\mathbf{F} = \mathbf{I} + k\mathbf{e}_1\mathbf{e}_2$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. It follows that

$$\mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_2\mathbf{e}_2, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = k\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2, \quad \mathbf{E} = \frac{k^2}{2}\mathbf{e}_2\mathbf{e}_2 + \frac{k}{2}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1). \tag{2.88}$$

and, consequently,

$$\mathbf{E} : \mathcal{A}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = A_2 \frac{k^2}{2} + A_6 \frac{k}{2}, \tag{2.89}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{A}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = A_2 k^2 + A_6 k + \frac{A_4}{4}. \tag{2.90}$$

Therefore, inequality (2.78)-(2.79) takes the form

$$q(k) := 6(A_2k^2 + A_6k) + A_4 \geq 0. \tag{2.91}$$

Taking k very large in (2.91) implies that $A_2 > 0$. Next, let consider $q''(k)$ and note that

$$q''(k) = 12A_2 > 0 \quad \forall k. \tag{2.92}$$

Thus $q(k)$ is a strictly convex function in $(-\infty, \infty)$ and, hence, it has a unique minimum. To evaluate the extreme k_{min} of $q(k)$, we consider

$$q'(k) = 6(2A_2k + A_6). \tag{2.93}$$

Immediately, it follows that

$$q'(k) = 0 \quad \text{at} \quad k_{min} = -\frac{A_6}{2A_2}. \tag{2.94}$$

Furthermore, the minimum of $q(k)$, q_{min}, is given by

$$q_{min} = q(k_{min}) = -\frac{3A_6^2}{2A_2} + A_4. \tag{2.95}$$

In virtue of (2.91), we derive (2.75).

Theorem 2.2.4 *If the Fung model (2.34) with $Q = Q_2$ satisfies the Strong Ellipticity condition, the parameters A_1, A_2, A_3, A_4 are positive and satisfy*

$$\sqrt{A_2\left(A_3 + \frac{A_4}{2}\right)} > \frac{1}{2}(A_2 + A_3), \tag{2.96}$$

$$\sqrt{A_1\left(A_3 + \frac{A_4}{2}\right)} > \frac{1}{2}(A_1 + A_3), \tag{2.97}$$

$$A_4 > \frac{3A_5^2}{2A_1}, \tag{2.98}$$

$$A_4 > \frac{3A_6^2}{2A_2}. \tag{2.99}$$

Proof: It is proved along the lines indicated in the proof of theorem (2.2.3).

2.3 The Tensor " C "as Strain Measure

Our task through this section will be to relax restrictions on the material parameters of the Fung model (2.34) by using the right Cauchy-Green deformation tensor $\mathbf{C}$ as strain measure.

The strain energy function is expressed as

$$W = \frac{c}{2}(e^{\widetilde{Q}} - d), \tag{2.100}$$

with c positive constant and $\widetilde{Q}$ defined by

$$\widetilde{Q} = \mathbf{C} : \mathcal{B} : \mathbf{C}, \tag{2.101}$$

where $\mathcal{B}$ is a constant fourth order tensor and $d = \mathbf{I} : \mathcal{B} : \mathbf{I}$ is a constant .

Particularly, we consider two forms of the strain energy function (2.100) with $\widetilde{Q}$ defined by

$$\widetilde{Q}_1 = B_1 C_{11}^2 + B_2 C_{22}^2 + 2B_3 C_{11} C_{22} + B_4 C_{12}^2. \tag{2.102}$$

$$\widetilde{Q}_2 = B_1 C_{11}^2 + B_2 C_{22}^2 + 2B_3 C_{11} C_{22} + B_4 C_{12}^2 + 2B_5 C_{11} E_{12} + 2B_6 C_{22} C_{12}. \tag{2.103}$$

To provide restrictions on the material parameters B_1, B_2, B_3, B_4, B_5, and B_6 in (2.102)-(2.103) we impose the L-H condition to be verified.

2.3.1 Restrictions on Material Parameters of W with $\widetilde{Q} = \widetilde{Q}_1$

Let $\{\mathbf{e}_i\}$ be an orthonormal basis for $\mathbb{E}^3$. Then, $\{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\}$ is a basis for the space of the fourth-order tensors. Consider the strain energy function (2.100) with $\widetilde{Q} = \widetilde{Q}_1$. Hence, the fourth-order tensor $\mathcal{B}$ in (2.101) can be written as

$$\begin{aligned} \mathcal{B}_1 \;=\;& B_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 + B_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + B_3(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1) \\ &+ \frac{B_4}{4}(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1), \end{aligned} \tag{2.104}$$

and, therefore,

$$\widetilde{Q}_1 = \mathbf{C} : \mathcal{B}_1 : \mathbf{C}. \tag{2.105}$$

We use the L-H condition to derive restrictions on B_1, B_2, B_3, and, B_4 and we find that

Theorem 2.3.1 *If the Fung model (2.100) with $\widetilde{Q} = \widetilde{Q}_1$ satisfies the Legendre-Hadamard condition, then the parameters B_1, B_2 are non-negative and*

$$2B_3 + B_4 \geq 0. \tag{2.106}$$

Proof: By taking into account the symmetries of the fourth-order tensor $\mathcal{B}_1$, it can be easily showed [34] that $\forall\, \mathbf{H} \in \text{Lin}$

$$\mathbf{H} : \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} = 2c^{\widetilde{Q}_1}[(2\mathbf{C} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + (\mathbf{C} : \mathcal{B}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})]. \tag{2.107}$$

Hence, L-H condition is satisfied if and only if

$$(2\mathbf{C} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + \mathbf{C} : \mathcal{B}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} \geq 0 \quad \forall\, \mathbf{F} \in \text{ Lin}^+, \tag{2.108}$$

$$\forall\, \mathbf{H} = \mathbf{ab} \neq \mathbf{0} \quad \text{with} \quad \mathbf{a}, \mathbf{b} \in \mathbb{E}^3 \quad \text{such that} \quad \mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0. \tag{2.109}$$

Let set $\mathbf{H} = \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3\mathbf{e}_3$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Thus,

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \lambda_1\mathbf{e}_1\mathbf{e}_2, \quad \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_2\mathbf{e}_2, \quad \mathbf{C} = \lambda_1^2\mathbf{e}_1\mathbf{e}_1 + \lambda_2^2\mathbf{e}_2\mathbf{e}_2 + \lambda_3^2\mathbf{e}_3\mathbf{e}_3. \tag{2.110}$$

It follows that

$$\mathbf{C} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = 0, \tag{2.111}$$

$$\mathbf{C} : \mathcal{B}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = B_2^2\lambda_2^2 + B_3^2\lambda_1^2, \tag{2.112}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \frac{B_4}{4}{\lambda_1}^2. \tag{2.113}$$

Condition (2.108)-(2.109) holds for the previous $\mathbf{H}$ and $\mathbf{F}$ if

$$2(B_2\lambda_2^2 + B_3\lambda_1^2) + B_4\lambda_1^2 \geq 0. \tag{2.114}$$

Considering all the possible positive stretch ratios λ_1 and λ_2 implies that

$$B_2 \geq 0 \quad \text{and} \quad 2B_3 + B_4 \geq 0. \tag{2.115}$$

Similarly, we take $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3\mathbf{e}_3$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Then, it follows

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \lambda_2\mathbf{e}_2\mathbf{e}_1, \quad \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_1\mathbf{e}_1, \quad \mathbf{C} = \lambda_1^2\mathbf{e}_1\mathbf{e}_1 + \lambda_2^2\mathbf{e}_2\mathbf{e}_2 + \lambda_3^2\mathbf{e}_3\mathbf{e}_3. \tag{2.116}$$

Hence,

$$\mathbf{C} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = 0, \tag{2.117}$$

$$\mathbf{C} : \mathcal{B}_1 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = B_1\lambda_1^2 + B_3\lambda_2^2, \tag{2.118}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \frac{B_4}{4}\lambda_2^2. \tag{2.119}$$

By the previous choices of $\mathbf{H}$ and $\mathbf{F}$, condition (2.108)-(2.109) becomes

$$2(B_1\lambda_1^2 + B_3\lambda_2^2) + B_4\lambda_2^2 \geq 0. \tag{2.120}$$

Since inequality (2.120) must be satisfied for all possible stretch ratios, we conclude

$$B_1 \geq 0 \quad \text{and} \quad 2B_3 + B_4 \geq 0. \tag{2.121}$$

Theorem 2.3.2 *If the Fung model (2.100) with $Q = Q_2$ satisfies the Strong Ellipticity condition then $B_1 > 0$, $B_2 > 0$, and $2B_3 + B_4 > 0$.*

Proof: The proof follows by the same arguments used in theorem (2.3.1).

2.3.2 Restrictions on Material Parameters of W with $\widetilde{Q} = \widetilde{Q}_2$

Recall that $\{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\}$ defines an orthonormal basis of the space of fourth-order tensors. Consider the strain energy function (2.100) with $\widetilde{Q} = \widetilde{Q}_2$. Then, the fourth-order tensor $\mathcal{B}$ in (2.101) can be written as

$$\begin{aligned} \mathcal{B}_2 \;=\; & B_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 + B_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + B_3(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1) + \frac{B_4}{4}(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 \\ & + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1) + \frac{B_5}{2}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1) \\ & + \frac{B_6}{2}(\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2). \end{aligned} \tag{2.122}$$

Thus,

$$\widetilde{Q}_2 = \mathbf{C} : \mathcal{B}_2 : \mathbf{C}. \tag{2.123}$$

Restrictions on B_1, B_2, B_3, B_4, B_5, and B_6 are obtained by imposing the L-H condition to be satisfied. We establish the following theorem.

Theorem 2.3.3 *If the Fung model (2.100) with* $\widetilde{Q} = \widetilde{Q}_2$ *satisfies the Legendre-Hadamard condition, then the parameters* B_1, B_2 *are non-negative and*

$$2B_3 + B_4 \geq 0, \tag{2.124}$$

$$2(B_1 + B_3) + B_4 \geq \frac{3B_5^2}{2B_1}, \tag{2.125}$$

$$2(B_2 + B_3) + B_4 \geq \frac{3B_6^2}{2B_2}. \tag{2.126}$$

Proof: By using the symmetries of the fourth-order tensor $\mathcal{B}_2$, it can be proved [34] that $\forall\, \mathbf{H} \in$ Lin

$$\mathbf{H} : \frac{\partial \hat{\mathbf{P}}_A}{\partial \mathbf{F}}(\mathbf{F}) : \mathbf{H} = 2c^{\widetilde{Q}_2}[(2\mathbf{C} : \mathcal{B}_1 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})^2 + (\mathbf{C} : \mathcal{B}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H})], \tag{2.127}$$

Therefore, a necessary condition for the L-H condition to be satisfied is the following

$$\mathbf{C} : \mathcal{B}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} + 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} \geq 0 \quad \forall\, \mathbf{F} \in \text{ Lin}^+, \tag{2.128}$$

$$\forall\, \mathbf{H} = \mathbf{ab} \neq \mathbf{0} \quad \text{with} \quad \mathbf{a}, \mathbf{b} \in \mathbb{E}^3 \quad \text{such that} \quad \mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0. \tag{2.129}$$

As in theorem (2.3.1), by choosing $\mathbf{H} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \lambda_1\mathbf{e}_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3\mathbf{e}_3$, we obtain that B_1, B_2 are nonnegative and $2B_3 + B_4 \geq 0$.

Next, let consider $\mathbf{H} = \mathbf{e}_2\mathbf{e}_1$ and $\mathbf{F} = \mathbf{I} + k\mathbf{e}_2\mathbf{e}_1$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. It can be easily derived that

$$\mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_1\mathbf{e}_1, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = k\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1, \quad \mathbf{C} = (1 + k^2)\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 + k(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) \tag{2.130}$$

Consequently,

$$\mathbf{C} : \mathcal{B}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = B_1 k^2 + B_5 k + B_1 + B_3, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = B_1 k^2 + B_5 k + \frac{B_4}{4}. \tag{2.131}$$

Hence, for the above choices of $\mathbf{F}$ and $\mathbf{H}$ condition (2.128)-(2.129) becomes

$$r(k) := 6(B_1 k^2 + B_5 k) + 2(B_1 + B_3) + B_4 \geq 0 \quad \forall k. \tag{2.132}$$

By taking k very large in (2.132), we deduce that $B_1 > 0$. Then, evaluate $r''(k)$ and note that

$$r''(k) = 12 B_1 > 0 \quad \forall k. \tag{2.133}$$

The function $r(k)$ is strictly convex in $(-\infty, +\infty)$ and, therefore, it has a unique minimum. To calculate the extreme k_{min} of $r(k)$, we consider

$$r'(k) = 6(2 B_1 k + B_5). \tag{2.134}$$

Thus, we obtain

$$r'(k) = 0 \quad \text{at} \quad k_{min} = -\frac{B_5}{2 B_1}. \tag{2.135}$$

The minimum of $r(k)$, r_{min}, is

$$r_{min} = r(k_{min}) = -\frac{3 B_5^2}{2 B_1} + 2(B_1 + B_3) + B_4. \tag{2.136}$$

Therefore, inequality (2.132) and (2.136) implies (2.125).

Next, let take $\mathbf{H} = \mathbf{e}_1 \mathbf{e}_2$, $\mathbf{F} = \mathbf{I} + k \mathbf{e}_1 \mathbf{e}_2$ so that $\mathbf{F}^{-\mathrm{T}} : \mathbf{H} = 0$. Thus,

$$\mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_2 \mathbf{e}_2, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = \mathbf{e}_1 \mathbf{e}_2 + k \mathbf{e}_2 \mathbf{e}_2, \quad \mathbf{C} = \mathbf{e}_1 \mathbf{e}_1 + (1 + k^2) \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3 + k(\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1) \tag{2.137}$$

Hence,

$$\mathbf{C} : \mathcal{B}_2 : \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H} = B_2 k^2 + B_6 k + B_2 + B_3, \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} : \mathcal{B}_2 : \mathbf{F}^{\mathrm{T}} \cdot \mathbf{H} = B_2 k^2 + B_6 k + \frac{B_4}{4}. \tag{2.138}$$

Condition (2.108)-(2.109) takes the form

$$s(k) := 6(B_2 k^2 + B_6 k) + 2(B_2 + B_3) + B_4 \geq 0 \quad \forall k. \tag{2.139}$$

Taking k very large in (2.139) yields $B_2 > 0$. Let consider $s''(k)$ and remark that

$$s''(k) = 12B_2 > 0 \quad \forall k. \tag{2.140}$$

We conclude that the function $s(k)$ is strictly convex in $(-\infty, +\infty)$ and, therefore, it has an unique minimum. To find it, we consider

$$s'(k) = 6(2B_2k + B_6). \tag{2.141}$$

Then, it follows that

$$s'(k) = 0 \quad \text{at} \quad k_{min} = -\frac{B_6}{2B_2}. \tag{2.142}$$

The minimum of $s(k)$, s_{min}, is given by

$$s_{min} = s(k_{min}) = -\frac{3B_6^2}{2B_2} + 2(B_2 + B_3) + B_4. \tag{2.143}$$

Inequality (2.126) follows from (2.139)-(2.143).

2.4 Conclusions

Although phenomenological equations have been widely adopted to describe the mechanical response of anisotropic biological materials, only recently some efforts have been invested to assess those equations by using constitutive assumptions [34].

We briefly presented some *order-preserving* conditions for constitutive equations and their physical implications. Particularly, by invoking the S-E and the L-H inequalities, we found necessary conditions on the material parameters of some forms of the exponential Fung model used for bovine pericardium. We showed that if the parameters do not meet those conditions, the constitutive model predicts a physically unreasonable behavior.

The restrictions on the material parameters result useful in fitting constitutive relations to the experimental data, besides validate the model itself. Thus, phenomenological laws gain statical properties still preserving their attractive mathematical simplicity.

This work should stimulate further interest in adopting physically sound constitutive assumptions to empirical laws. However, since setting bounds on the material parameters in some models is not always an easy task, a *structural* approach is preferable.

3.0 A STRUCTURAL MODEL

Phenomenological constitutive models are formulated by regression of experimental data according to the fundamental principles of continuum mechanics. They are not based on any physical reasoning and they lack any relation to the tissues structure. Our first goal was to attribute physical meaning to the parameters of some phenomenologial laws used to model bovine pericardium. By employing the L-H and S-E conditions, we obtained some necessary conditions for those parameters to be satisfied. Derivation of sufficient conditions needed more complicated mathematical studies and, therefore, it has not been treated in this study.

Due to difficulties to associate the parameters with the mechanical and morphological properties of the tissue in phenomenological models, structural models have been preferred. They are formulated on the basis of the observed structural and mechanical features of the constituents of the tissue.

We present a structurally based constitutive model for bovine pericardium, which extends works by Billiar and Sacks [2, 19]. The mechanical response of the tissues is attributed solely to collagen fibers. The matrix contribution is neglected. Viscous components of the tissue are not considered in this formulation. Fibers are assumed to be arranged spatially according to a Gaussian distribution. Each fiber appears undulated in the reference configuration and it deforms linearly. Upon stretch, it becomes taut and, successively, it starts to bear load according to a recruitment function, which is defined by a Gamma probability density function.

The model accounts for the material non-linearity and anisotropy. It includes the description of the failure process. Four structurally based parameters need to be determined. The model is fitted with available data on bovine pericardium.

3.1 Previous Works

We present some structural constitutive models which have been used for collagenous tissues.

Hurschler et al. [12] proposed a constitutive model for tendons and ligaments by taking into consideration their structural and microstructural properties.

In this presentation Kastaelic's fiber hierarchical organization is adopted [13]. The tissue is assumed to be made of fascicles, fascicle are aggregations of fibers, and, fibers are composed of collagen fibrils.

Consider a volume of tissue. It is assumed that fibers are at different levels of crimp in the undeformed configuration and they are predominantly oriented longitudinally. Let l_o be the length of the fiber in the reference configuration, l be the length of the deformed fiber and l_s the length of the fiber which straightens and begins to bear load. Consequently, the tissue stretch-ratio is $\lambda_t = \frac{l}{l_o}$, the straightening stretch-ratio (SSR) of a fiber, defined as the stretch at which the fiber straightens and begins to bear load, is $\lambda_s = \frac{l_s}{l_o}$. Fibers are assumed to deform according to the constitutive relation $\sigma_{33}(\lambda_3)$ where σ_{33} is the axial normal stress and λ_3 is the axial stretch-ratio of the fiber. Thus, the stress in the tissue is obtained by summing the contributions of each fibers over the cross section of the tissue volume.

A Weibull probability density function is used to describe the distribution of SSR of the fibers in the tissue volume. It has the following form

$$P_w(\lambda_s) = \frac{\beta}{\delta}(\frac{\lambda_s - \gamma}{\delta})^{\beta-1} e^{[-(\frac{\lambda_s-\gamma}{\delta})^{\beta}]}, \qquad \lambda_s > \gamma \tag{3.1}$$

$$P_w(\lambda_s) = 0, \quad \lambda_s \leq \gamma \tag{3.2}$$

where β ($\beta > 0$), δ ($\delta > 0$) and γ ($\gamma > 0$ since $\lambda > 0$) are the shape, scale, and location parameters. The Weibull PDF has the advantage of being one-tailed and, therefore, the probability that λ_s is less than γ is zero.

The overall tissue deformation λ_t and the states of deformation of the population of the fibers, which constitutes the tissue according to the SSR distribution $P_w(\lambda_s)$, determine the stress in the tissue volume. Thus, the longitudinal normal stress in a ligament or a tendon is given by

$$\sigma_t(\lambda_t) = \int_{\gamma}^{\lambda_t} P_w(\lambda_s)\sigma_{33}(\frac{\lambda_t}{\lambda_s})\, d\lambda_s, \quad \lambda_t > \gamma \tag{3.3}$$

and, since $P_w(\lambda_s) = 0$ when $\lambda_s \leq \gamma$,

$$\sigma_t(\lambda_t) = 0, \quad \lambda_t \leq \gamma \tag{3.4}$$

where λ_t is the stretch-ratio of the ligament and $\sigma_{33}(\frac{\lambda_t}{\lambda_s})$ is the longitudinal normal stress of the fiber with SSR λ_s.

When $\lambda_t \leq \gamma$, the fibers are unloaded. Furthermore, when $\gamma < 1$, some fibers are loaded in the reference configuration and, when $\gamma \geq 1$, fibers bear load only after they are stretched of an amount γ.

Hurschler et al. [12] proposed a microstructural model for the fiber constitutive law σ_{33}.

Collagenous tissue is assumed to be an aggregation of collagen fibrils embedded in a proteoglycan matrix. The constitutive relation for a fiber is expressed in terms of a strain energy function. Fibrils are assumed to deform uniaxially in various orientations in three dimensions and they contribute to the strain energy when loaded. The matrix contributes to the stress throughout the hydrostatic pressure p. Assuming that fibers undergo to an homogeneous deformation described by the gradient of deformation $\mathbf{F}$

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{3.5}$$

the stress components per unit area in the deformed configuration are expressed by:

$$\sigma_{ij} = \frac{\lambda_i \lambda_j}{\mid J \mid} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} v_f R_{ij}(\phi,\theta) \frac{f(\lambda)}{\lambda}\, d\phi\, d\theta + p\delta_{ij} \tag{3.6}$$

where $R(\phi, \theta)$ is the fibrils orientation and ϕ and θ are the longitude and the colatitude angles of a spherical coordinate system, respectively; $\mid J \mid$ is the Jacobian of the deformation; λ is the fibril stretch-ratio; $f(\lambda)$ is the fibril constitutive law; v_f is the fibril volume fraction; δ_{ij} is the Kronecker delta.

The fibril orientation is represented by the following probability density function:

$$R(\phi, \theta) = R(\theta) = \frac{k \cosh(k \cos \phi)}{2\pi \sinh k} \tag{3.7}$$

with $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$ and where k is the concentration parameter. The orientation distribution is symmetric about the x_3 axis and it is defined such that

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} R(\phi, \theta) \sin \phi \, d\phi \, d\theta = 1. \tag{3.8}$$

The contribution of the pressure to the total stress due to the matrix is assumed to be negligible. Fiber is assumed to undergo incompressible axisymmetric deformation about the x_3-axis. Therefore,

$$J = \lambda_1 \lambda_2 \lambda_3 = 1 \quad \text{and} \quad \lambda_1 = \lambda_2 = \frac{1}{\sqrt{\lambda_3}}. \tag{3.9}$$

The fibril is assumed to bear load when stretched and to deform linearly, i.e.

$$f(\lambda) = E_f(\lambda - 1) \quad \text{when} \quad \lambda > 1, \tag{3.10}$$

$$f(\lambda) = 0 \quad \text{when} \quad \lambda \leq 1, \tag{3.11}$$

where $f(\lambda)$ is the load per unit area and E_f is the fibril elastic modulus. Furthermore, the fibril stretch ratio in the ϕ direction

$$\lambda = \sqrt{{\lambda_3}^2 \cos^2 \phi + \frac{1}{\lambda_3} \sin^2 \phi} \tag{3.12}$$

Substituting $R_{33}(\phi, \theta) = R(\phi, \theta) \sin \phi \cos^2 \phi$, Eqs. (3.7)-(3.9)-(3.10)-(3.11) into Eq.(3.6) gives the following fiber constitutive law for normal stress in the x_3-direction:

$$\sigma_{33}(\lambda_3) = 2\pi \lambda_3^2 v_f E_f \int_0^{\frac{\pi}{2}} R(\phi) \left(1 - \frac{1}{\lambda}\right) \sin \phi \cos^2 \phi \, d\phi \tag{3.13}$$

where $R(\phi)$ is given by Eq. (3.7). Finally, the tissue stress-stretch relation is obtained by using Eq. (3.13) into Eqs. (3.3)-(3.4).

The model describes also the failure process included either at the fiber scale either at the fibril scale. At the structural level, each fiber is assumed to fail when the stretch ratio exceeds a certain value λ_f. Furthermore, once the fibers fail, they do no longer bear load. Thus, the stress-stretch relations (3.3)-(3.4) take the form

$$\sigma_t(\lambda_t) = \int_{\gamma}^{\lambda_t} P_w(\lambda_s)\sigma_{33}(\frac{\lambda_t}{\lambda_s})\, d\lambda_s, \quad (\frac{\lambda_t}{\lambda_f} \leq \gamma \quad \text{and} \quad \lambda_t > \gamma) \tag{3.14}$$

$$\sigma_t(\lambda_t) = \int_{\frac{\lambda_t}{\lambda_f}}^{\lambda_t} P_w(\lambda_s)\sigma_{33}(\frac{\lambda_t}{\lambda_s})\, d\lambda_s, \quad (\frac{\lambda_t}{\lambda_f} > \gamma \quad \text{and} \quad \lambda_t > \gamma) \tag{3.15}$$

$$\sigma_t(\lambda_t) = 0, \quad (\lambda_t \leq \gamma) \tag{3.16}$$

At the microstructural level, the fibril fails when its stretch-ratio exceeds a certain λ_f. This failure law can be included into to the model (3.13) by mean of a unit step function $\delta(\lambda - \lambda_f)$. Thus, the fiber constitutive law (3.13) assumes the form

$$\sigma_{33}(\lambda_3) = 2\pi\lambda_3^2 v_f E_f \int_0^{\frac{\pi}{2}} R(\phi)\left(1 - \frac{1}{\lambda}\right)\delta(\lambda - \lambda_f)\sin\phi\cos^2\phi\, d\phi \tag{3.17}$$

The tissue stress-stretch relation is obtained by substituting the fiber constitutive law into Eqs. (3.3)-(3.4).

By assuming the linear stress-stretch relation for fiber, the number of parameters into the model has been reduced to five (α, β, γ, k, and λ_f). Experimental data have been fitted by the model resulting in a very good fit.

Liao and Belkoff [17] have developed a simple structural model for ligaments by assuming the sequential uncrimping and stretching of the collagen fibers.

Let N be an unknown number of collagen fibers in the soft tissue. Assume that fibers are undulated in the stress-free configuration and they trasmit load when they are straightened. The initial lengths of the fibers are assume to be randomly distributed and a Gaussian distribution with mean μ and standard deviation σ describe the normalized initial lengths of the fibers which is defined as the initial length divided by the ligament gage length. Then, the number of fibers straightened at an intermediate stretch, x, is given by

$$dN(x) = N\frac{1}{\sqrt{2\pi}s}e^{-\frac{(x-\mu)^2}{2s^2}}\, dx. \tag{3.18}$$

At a stretch λ, the fibers recruited at x are characterized by $\frac{\lambda}{x} - 1$ strain. Each fiber is assumed to be linear elastic with modulus E_i and cross sectional area A_i. The axial force of the ligaments is the continuous sum of the axial force of each straightened fibers. Therefore,

$$F(\lambda) = \int_1^\lambda A_i E_i \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-\mu)^2}{2s^2}} \frac{\lambda - x}{x} dx. \tag{3.19}$$

Since N and A_i are unknown, an *effective modulus* E is defined such that $E = (A_i E_i N)/A$. Thus,

$$\sigma(\lambda) = \int_1^\lambda E \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-\mu)^2}{2s^2}} \frac{\lambda - x}{x} dx. \tag{3.20}$$

Failure is included into the model by assuming that fibers fail at the same limit strain α. Note that since the fibers are assumed to have the same elastic modulus and the same limit strain, they fail in the same sequence in which they are recruited. Thus, the number of fibers recruited at a stretch $\frac{x}{(1+\alpha)}$ is given by

$$dN\left(\frac{x}{1+\alpha}\right) = \frac{N}{\sqrt{2\pi s}} e^{-\frac{((\frac{x}{1+\alpha})-\mu)^2}{2s^2}} d\left(\frac{x}{1+\alpha}\right) \qquad (x \geq 1+\alpha). \tag{3.21}$$

The resulting stress at the stretch λ is expressed as

$$\sigma(\lambda) = \int_{\frac{\lambda}{1+\alpha}}^\lambda E \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-\mu)^2}{2s^2}} \frac{\lambda - x}{x} dx \qquad (\lambda \geq 1+\alpha) \tag{3.22}$$

and

$$\sigma(\lambda) = \int_1^{\frac{\lambda}{1+\alpha}} E \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-\mu)^2}{2s^2}} \frac{\lambda - x}{x} dx \qquad (\lambda \leq 1+\alpha). \tag{3.23}$$

The model, which contains only four structural parameters, was able to describe the behavior of rabbit Medial Collateral Ligaments (MCLs) which fail abruptly better than the behavior of MCLs which presented a prolonged failure response.

Due maybe to the fact that fiber-fiber and fiber-matrix interactions are not taken into account, the model could not simulate the post failure behavior. Furthermore, the strain is assumed to be homogeneous along the length of the ligaments. Fibrils deformations are not considered in order not to add further parameters to be determined.

Kwan and Woo [35] suggest a structurally based model for collagenous tissues in which the collagen fibrils are arranged in a parallel manner.

The mechanical response of parallel-collagenous tissues subjected to uniaxial tension has been found to be nonlinear by many investigators. The gradual uncrimping of the fibrils is responsible for the initial low modulus region. It follows a region of gradually increasing modulus due to the resistance of the fibrils to the load. Once all the fibrils become taut, a maximum modulus region is achieved and the stress increases linearly with the strain. A region of decreasing modulus describes the failure of groups of fibrils until complete tissue failure occurs.

Since the crimping of a fibril can be removed by applying a small tension, the stress-strain relationship can be assumed to be linear until the fibril becomes taut. After that, a higher stress is required to strain the fibril. This suggested the assumption of a bilinear constitutive law for a single fibril. Let σ and ϵ be the tensile stress and strain, E_s the initial modulus, E_e the elastic modulus with $E_e > E_s$, ϵ_s the uncrimping strain and ϵ_u the ultimate strain. Thus, the stress-strain relation for a single fibril can be expressed as

$$\sigma = \begin{cases} E_s\epsilon, & 0 < \epsilon \leq \epsilon_s; \\ E_s(\epsilon - \epsilon_s) + E_s\epsilon_s, & \epsilon_s < \epsilon < \epsilon_u. \end{cases} \tag{3.24}$$

Collagen tissue is composed of fibrils of different lengths in the slack configuration. Thus, they become taut at different strains. The tissues is assumed to be made of m groups of fibrils with different lengths γ_1, γ_2,...,γ_m (where $\gamma_1+\gamma_2+...+\gamma_m = 1$). Let ϵ_{s1}, ϵ_{s2},...,ϵ_{sm} be the strains at which each group of fibrils become taut. All the fibrils are taut at ϵ_{sm}, when the maximum tissues modulus is reached. Thereafter, they begin to fail. The fibrils are divided into n failure groups β_1, β_2,...,β_n (where $\beta_1+\beta_2+...+\beta_n=1$) which fail at strain ϵ_{u1}, ϵ_{u2},..., ϵ_{un}, respectively. A $m \times n$ nth order stress-strain relation is obtained for the tissue by using the fibril relation (3.24) with the superposition principle.

A 3×3 model has been used to fit tensile stress-strain data obtained form rabbit anterior cruciate ligaments (ACL) and canine medial collateral ligaments (MCL). The model can describe the mechanical properties of parallel-fibered collagenous tissues. It determines the percentages of fibrils and the strains at which they become taut as well as the percentages of the fibrils and the strains at which they fail.

3.2 Model Formulation

The present constitutive model is a generalization of previous models for collagenous tissues presented by Billiar and Sacks [2, 19].

Bovine pericardium is a fibrous membrane composed of mainly collagen and elastin fibers embedded in a ground substance called *matrix*. In the our formulation, we ignore the mechanical contribution of the elastin fibers and we assume that the hydrostatic forces due to the matrix are negligible.

We model bovine pericardium as an anisotropic, incompressible and hyperelastic material [27]. The viscoelastic aspect of the tissue response is not considered in the present study. Thus, using the first Piola-Kirchhoff stress tensor, the constitutive equation takes the form [28]

$$\mathbf{P} = -p\mathbf{F}^{-\mathrm{T}} + 2\mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{C}}, \tag{3.25}$$

where p is a pressure, $\mathbf{F}$ is the deformation gradient tensor, $\mathbf{F}^{\mathrm{T}}$ is its transpose, $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ is the right Cauchy-Green deformation tensor and $W = W(\mathbf{C})$ is the strain energy function per unit volume.

The energy stored by the tissue during a deformation is assumed to be solely due to the fiber extension and, hence, shear and bending energies are neglected. Therefore, the strain energy function of the tissue's fibers network can be expressed as [16]

$$W = \int_{\Sigma} R(\mathbf{n}) w_f[\lambda(\mathbf{C}, \mathbf{n})] dA, \tag{3.26}$$

where $\mathbf{n}$ is the material direction in the reference configuration, Σ is a unit hemisphere with outward normal $\mathbf{n}$, $w_f(\lambda)$ is the strain energy per unit volume corresponding to an axial fiber stretch λ which is defined as

$$\lambda(\mathbf{C}, \mathbf{n}) = \sqrt{\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n}}, \tag{3.27}$$

and $R(\mathbf{n})$ is the probability density function for fibers whose mean axis is parallel to $\mathbf{n}$ with

$$\int_{\Sigma} R(\mathbf{n}) dA = 1. \tag{3.28}$$

Thus, $R(\mathbf{n})$ defines the predominant direction in which collagen fibers are arranged into the tissue.

Consequently, the first Piola-Kirchhoff stress tensor is

$$\mathbf{P} = -p\mathbf{F}^{-\mathrm{T}} + \int_\Sigma \frac{R(\mathbf{n})}{\lambda(\mathbf{C},\mathbf{n})}\sigma_f[\lambda(\mathbf{C},\mathbf{n})]\,\mathbf{F}\cdot\mathbf{nn}\,dA, \tag{3.29}$$

where $\sigma_f = w_f'(\lambda)$ is the fiber stress-stretch relation.

We assume that collagen fibers, which appear undulated in the stress-free configuration, support load only when they become taut (Fig.(3.1)). This assumption is reasonable since the load necessary to straighten collagen fibers is negligible. Thus, the strain energy function admits the following representation

$$w_f(\lambda) = \int_1^\lambda P(\lambda_s)\hat{w}_f\Big(\frac{\lambda}{\lambda_s}\Big)d\lambda_s; \quad \hat{w}_f(1) = 0, \tag{3.30}$$

where $\hat{w}_f(\lambda)$ is the strain energy per unit volume for a taut fiber stretched of an amount λ, $P(\lambda_s)$ is the probability density function for fibers that become taut at a stretch λ_s and $\frac{\lambda}{\lambda_s}$ is the stretch with respect to a taut configuration that occurs at a stretch λ_s. The resulting fiber stress-stretch relation can be expressed as

$$\sigma_f(\lambda) = w_f'(\lambda) = \int_1^\lambda \frac{1}{\lambda_s}P(\lambda_s)\hat{\sigma}_f\Big(\frac{\lambda}{\lambda_s}\Big)d\lambda_s, \tag{3.31}$$

where $\hat{\sigma}_f(\lambda) = \hat{w}_f'(\lambda)$.

In order to include failure in the previous model, each collagen fiber is assumed to fail at the same stretch λ_f. Therefore, the stress-stretch relation $\hat{\sigma}_f$ is replaced by

$$\tilde{\sigma}_f(\lambda) = \begin{cases} \hat{\sigma}_f(\lambda) & \text{if } \lambda < \lambda_f, \\ 0 & \text{if } \lambda \geq \lambda_f. \end{cases} \tag{3.32}$$

It follows that the overall fiber stress-stretch relation can be expressed as

$$\sigma_f(\lambda) = \int_1^\lambda \frac{1}{\lambda_s}P(\lambda_s)\tilde{\sigma}_f\Big(\frac{\lambda}{\lambda_s}\Big)d\lambda_s, \tag{3.33}$$

or, equivalently,

$$\sigma_f(\lambda) = \int_{\frac{\lambda}{\lambda_f}}^\lambda \frac{1}{\lambda_s}P(\lambda_s)\hat{\sigma}_f\Big(\frac{\lambda}{\lambda_s}\Big)d\lambda_s, \quad \text{when} \quad \lambda \leq \lambda_f \tag{3.34}$$

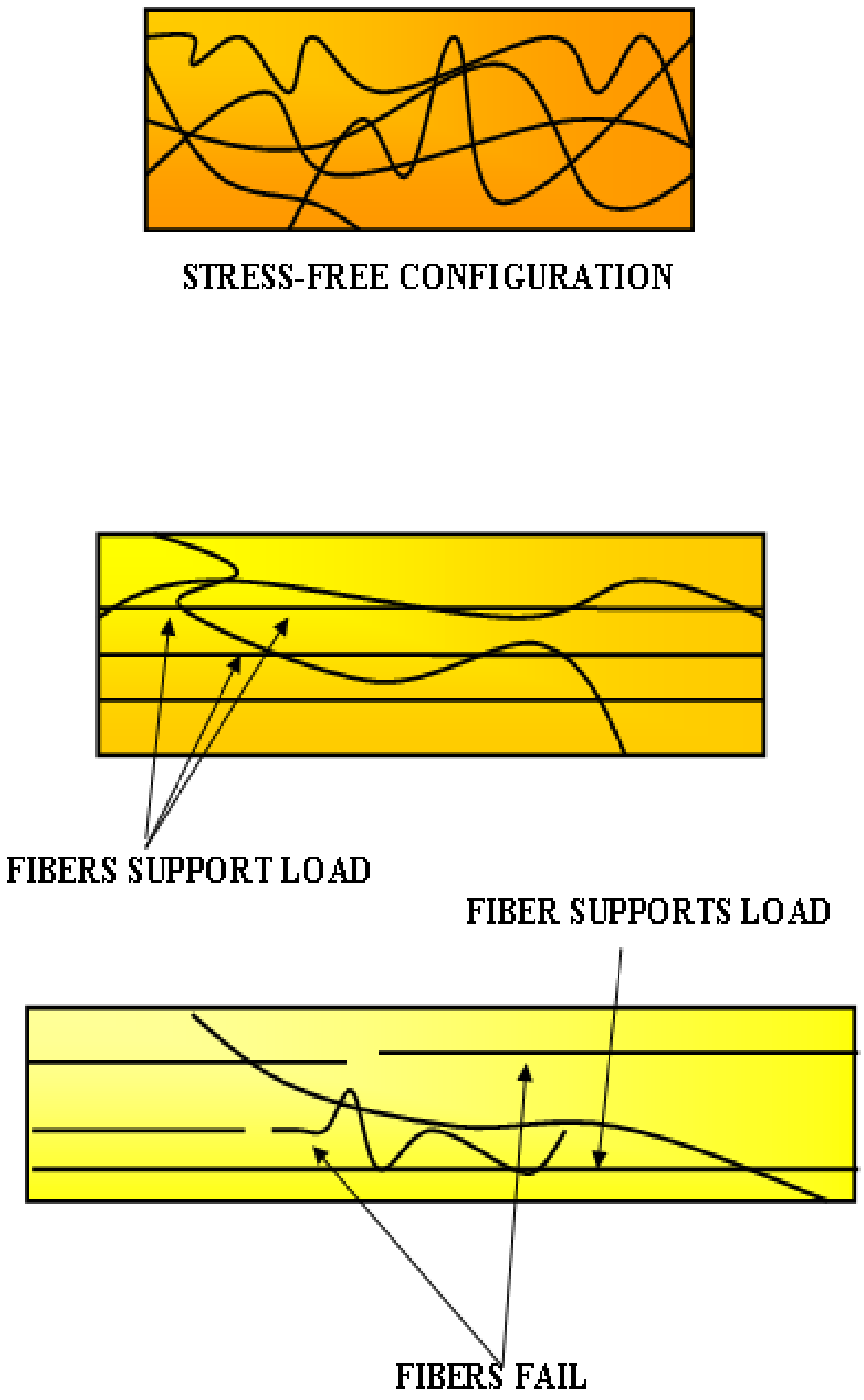

Figure 3.1. Assumptions on Collagen Fibers.

and

$$\sigma_f(\lambda) = \int_1^{\lambda} \frac{1}{\lambda_s} P(\lambda_s) \hat{\sigma}_f\left(\frac{\lambda}{\lambda_s}\right) d\lambda_s, \quad \text{when} \quad \lambda < \lambda_f. \tag{3.35}$$

Substituting Eq. (3.33) or Eqs. (3.34)-(3.35) into Eq. (3.29) gives the total stress $\mathbf{P}$.

3.3 Uniaxial Extension

To analyze the capabilities of the model, we consider the following incompressible planar deformation,

$$x_1 = \lambda_1 X_1, \quad x_2 = X_2, \quad x_3 = \frac{1}{\lambda_1} X_3, \tag{3.36}$$

where X_A and x_i are locations of material particles in the undeformed and deformed configurations, respectively, and λ_1 is the stretch ratio in the x_1 direction. Therefore, the deformation gradient $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ has the matrix form

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda_1} \end{pmatrix}, \tag{3.37}$$

and, consequently, the right Cauchy-Green deformation tensor $\mathbf{C}$ in matrix form is given by

$$\mathbf{C} = \begin{pmatrix} {\lambda_1}^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{{\lambda_1}^2} \end{pmatrix}. \tag{3.38}$$

Collagen fibers are assumed to be all oriented in on a thin plane subjected to a plane-stress deformation. This assumption yields

$$p = 0 \tag{3.39}$$

in Eq. (3.29).

From Eqs. (3.39), (3.27) and (3.29) with $\mathbf{n} = (\cos\theta, \sin\theta, 0)$, the nonzero components of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ are

$$P_{11} = \lambda_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{R(\theta)}{\lambda(\lambda_1, \theta)} \sigma_f(\lambda(\lambda_1, \theta)) \cos^2\theta d\theta, \tag{3.40}$$

$$P_{22} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{R(\theta)}{\lambda(\lambda_1, \theta)} \sigma_f(\lambda(\lambda_1, \theta)) \sin^2\theta d\theta, \tag{3.41}$$

$$P_{12} = \lambda_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{R(\theta)}{\lambda(\lambda_1, \theta)} \sigma_f(\lambda(\lambda_1, \theta)) \cos\theta \sin\theta d\theta, \tag{3.42}$$

$$P_{21} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{R(\theta)}{\lambda(\lambda_1, \theta)} \sigma_f(\lambda(\lambda_1, \theta)) \sin\theta \cos\theta d\theta \tag{3.43}$$

with

$$\lambda = \sqrt{{\lambda_1}^2 \cos^2\theta + \sin^2\theta}. \tag{3.44}$$

The function $R(\theta)$ can be choosen to be a Gaussian distribution

$$R(\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\theta-M)^2}{(2\sigma^2)}} \tag{3.45}$$

where σ is the standard deviation and M is the mean of the distribution. By using small angle light scattering technique [23], the function $R(\theta)$ can be determined experimentally. For bovine pericardium, the values of σ and M have been found to be equal to 35° and 0, respectively [21].

The fiber stress-stretch relation is assumd to be linear, i.e.

$$\hat{\sigma}_f(\lambda) = K(\lambda - 1), \tag{3.46}$$

where K is stiffness fiber costant [24]. Furthermore, we choose the crimp probability density function in the form of a Gamma distribution

$$P(\lambda_s) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \big(\lambda_s - 1\big)^{\alpha-1} e^{-\frac{(\lambda_s - 1)}{\beta}}, \tag{3.47}$$

with α and β parameters. $P(\lambda_s)$ is one-tailed and satisfies

$$\int_1^\infty P(\lambda_s)\, d\lambda_s = 1. \tag{3.48}$$

Then, from Eqs. (3.46)-(3.47), we obtain the following fiber stress-stretch relation becomes

$$\sigma_f(\lambda) = \frac{K}{\beta^\alpha \Gamma(\alpha)} \int_{\frac{\lambda}{\lambda_f}}^{\lambda} \frac{(\lambda - \lambda_s)}{{\lambda_s}^2} (\lambda_s - 1)^{\alpha-1} e^{-\frac{(\lambda_s-1)}{\beta}} d\lambda_s \quad \text{when} \quad \lambda \leq \lambda_f, \tag{3.49}$$

and,

$$\sigma_f(\lambda) = \frac{K}{\beta^\alpha \Gamma(\alpha)} \int_1^{\lambda} \frac{(\lambda - \lambda_s)}{{\lambda_s}^2} (\lambda_s - 1)^{\alpha-1} e^{-\frac{(\lambda_s-1)}{\beta}} d\lambda_s \quad \text{when} \quad \lambda > \lambda_f. \tag{3.50}$$

Substituting Eqs. (3.45)-(3.50) into Eqs. (3.40) gives the component P_{11} of the first Piola-Kirchhoff stress tensor:

$$P_{11} = \left(\frac{18\sqrt{2}}{7\pi^{\frac{3}{2}}}\right)\frac{K\lambda_1}{\beta^{\alpha}\Gamma(\alpha)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{\frac{\lambda}{\lambda_f}}^{\lambda}\left(\frac{\lambda-\lambda_s}{\lambda_s^2}\right)(\lambda_s-1)^{(\alpha-1)}e^{-\frac{(\lambda_s-1)}{\beta}}d\lambda_s\right\}e^{-\frac{648}{49}(\frac{\theta}{\pi})^2}\cos^2\theta d\theta, \tag{3.51}$$

when $\lambda \leq \lambda_f$, and

$$P_{11} = \left(\frac{18\sqrt{2}}{7\pi^{\frac{3}{2}}}\right)\frac{K\lambda_1}{\beta^{\alpha}\Gamma(\alpha)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{1}^{\lambda}\left(\frac{\lambda-\lambda_s}{\lambda_s^2}\right)(\lambda_s-1)^{(\alpha-1)}e^{-\frac{(\lambda_s-1)}{\beta}}d\lambda_s\right\}e^{-\frac{648}{49}(\frac{\theta}{\pi})^2}\cos^2\theta d\theta, \tag{3.52}$$

when $\lambda > \lambda_f$.

Thus, adopting the previous assumptions, the four parameters $\{\alpha, \beta, K, \lambda_f\}$ need to be evaluated in order to get the response of the material. Similarly, the remaning nonzero components of the first Piola-Kirchhoff stress tensor can be determined.

3.4 Structural Parameters

Uniaxial tests on bovine pericardium have been performed by Zioupous and Barbenel [36]. They revealed significant differences into the nominal stress-stretch ratio relations for strips of tissue at various orientations in the same pericardial sac. Particularly, strips of tissue aligned circumferentially with respect to the heart achieved a higher stress value followed by abrupt failure whereas strips aligned axially with respect to the heart are characterized by a lower stress value and a progressive failure. We use the experimental data obtained from uniaxial test of strip of pericardium in the circumferential direction to test the proposed model.

To determine the structural model parameters, the function

$$\mathcal{E} = \sum_{i=1}^{n} [P_{11}^{(i)} - P_{11}^{(i)}(\alpha, \beta, K, \lambda_f)]^2 \tag{3.53}$$

has been minimized by implementing a differential evolution code. The code has been provided by ICSI (International Computer Science Institute, Berkley, CA) and its use has been described by Price and Storn [18]. In Eq. (3.53), $P_{11}^{(i)}$ is the experimentally measured nominal stress at stretch $\lambda_1^{(i)}$; $P_{11}^{(i)}(\alpha, \beta, K, \lambda_f)$ is the theoretical nominal stress at stretch $\lambda_1^{(i)}$; n=44 is the number of data points used. The best fit values of the parameters are shown in Table (3.1) and the fitting curve is illustrated in Fig.(3.3). The agreement between experimental and theoretical nominal stress is reasonably good ($r^2 = 0.98$).

The model was able to describe qualitatively the tissue mechanical response. The theoretical nominal stress is underestimated (Fig.(3.3)). This may be due to the matrix and elastin fibers contributions, which have been not considered in our formulation.

According to our results, each fiber fails at stretch ratio $\lambda_f = 1.3$ and it deforms linearly with a stiffness parameter $K = 86$ MPa. Obviously, those assumptions can be removed to improve the model to the detriment of an increase of the number of parameters and, consequently, of computational time.

Table 3.1. Parameters set used in the model

parameter	value
α	36
β	$67 * 10^{-4}$
λ_f	1.3
K	86 MPa
r^2	0.98

The parameters α and β define how fast the fibers become straightened, start to support load, and, ultimately, fail. Fig.(3.2) shows the recruitment function for values of α and β in Table (3.1). The mean and the variance of $\lambda_s - 1$ are $\mu = \alpha\beta = 24 * 10^{-2}$ and $\sigma^2 = \alpha\beta^2 = 16 * 10^{-4}$, respectively.

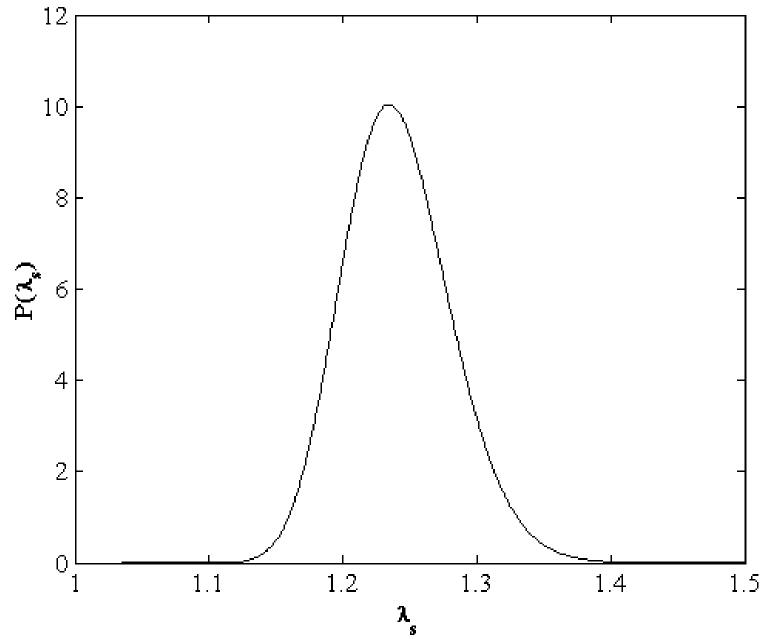

Figure 3.2. Crimp probability density function for α and β in Table (3.1).

3.5 Conclusions

We proposed a structural based constitutive equation for the description of the anisotropic mechanical response of bovine pericardium. The model was able to reproduce the toe region, the linear region throughout the failure region of the stress-stretch relation observed into an experimental investigation [36].

An estimation of the model parameters has been obtained by using a differential evolution code. However, since computation is time consuming, a faster fitting procedure will be developed in future.

The constitutive model presented has been successfully tested for some other soft biological tissues. Thus, our next goal will be to formulate a general structural constitutive model able to describe the mechanical behavior throughout failure of skin, aneurysms, tendons, ligaments, and other valvular tissues. Furthermore, since soft biological tissues are viscoelastic, we will take into consideration the time-dependent aspects of the tissues.

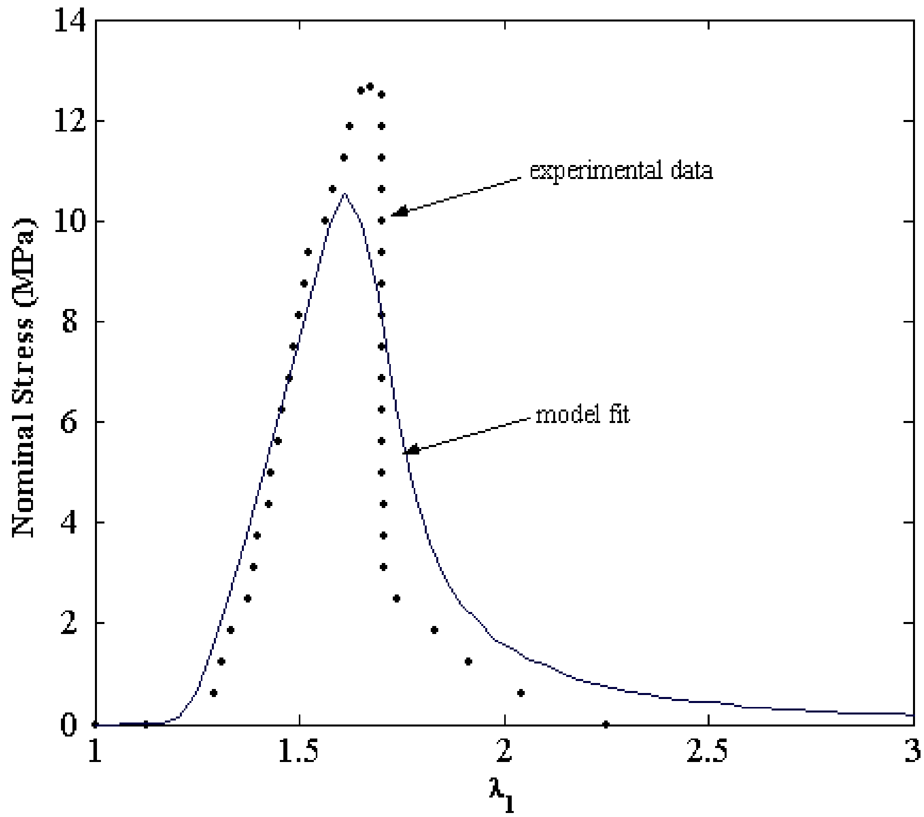

Figure 3.3. Model fit for strip of material aligned along the circumferential direction.

4.0 SUMMARY

In the first chapter, motivations for studying bovine pericardium have been briefly presented. This tissue has been used extensively to construct heart valves substitutes due to its excellent biocompatibility and low thrombogenicity. However, primary tissue's failure still limits the bovine pericardial heart valves' durability. Therefore, understanding the stress-strain behavior and the failure mode for this tissue is of fundamental importance. To this end, constitutive equations either of phenomenological kind either of structural kind need to be developed.

In the second chapter, the Strong Ellipticity and the Legendre-Hadamard inequalities are used to restrict the range of variability of parameters in a phenomenological constitutive equation, the exponential Fung law, employed to characterize the mechanical behavior of bovine pericardium[32]. Restrictions on the parameters, which are necessary for the S-E and L-H conditions, are derived for various forms of the model by applying Walton and Wilber's results [34].

In the third chapter a structural constitutive model for failure has been presented. The model is an extension of previous models proposed by Billiar and Sacks [2, 19]. The mechanical response of bovine pericardium has been assumed to be due to collagen fibers (matrix and elastin fibers' contributions are ignored). The angular distribution of the collagen fibers has been determined by using SALS. Only four structural parameters needed to be evaluated to completely characterize the mechanical response of the tissue when subjected to uniaxial load. The model has been fitted with success to published stress-strain data by implementing a differential evolution code.

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